Lifecycle Investing when House Prices are Cointegrated with Income

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Abstract

This paper studies the possibilities that residential real estate investment brings to lifecycle portfolios by incorporating the long-run relationship between house prices and income into the model. Labor income is at the core of lifecycle models and a significant body of literature argues that income is cointegrated with house prices through rents. Under this hypothesis, residential real estate investment can be used to hedge regional income changes on top of hedging rents. Regional income is very much related to the cost of labor intensive services such as elderly care, health care or education, and these services together with housing can constitute a sizable portion of household budgets. What makes investment in real estate attractive is thus the potential to hedge both housing and labor intensive services by exposing the portfolio to income changes. Preliminary results can partly rationalize the traditional role of housing, and not stocks, as the primary savings vehicle of households.

Suitability of lifelong financial plans can benefit from explicitly considering housing, labor income and the links between them. Housing represents about a fifth of household budgets, while labor income is typically the main source of inflows. Additionally, labor prices can relate to housing and other expense categories in as much as their production requires the use of labor, or their limited supply and widespread desirability result in equilibrium prices being driven by households' purchasing power and hence by labor income.

Understanding the dynamics of house prices is a challenge that has attracted the attention of many scholars. Early models sought to explain house prices in terms of fundamental economic variables like income growth, interest rates or constructions costs (Capozza & Helsley, 1989, 1990), but the exuberance of price fluctuations called into question the efficiency and rationality of house prices (Case & Shiller, 1989, 1990). Abraham and Hendershott (1996) attributed this mismatch to speculative bubbles and Malpezzi (1999) built an error correction model with random perturbations where prices slowly adjusted towards equilibrium prices. The equilibrium adjustment mechanism was formulated in terms of a stationary price-to-income ratio. The relevance of income comes from its role in determining equilibrium rent prices, as house prices can be understood as the value of discounted future rents.

In parallel, a purely financial model emerged that decomposed returns into news about future payoffs and news about future returns. Similar to the Gordon growth model, the price of a financial product is conceptualized as the present value of discounted future payoffs, but it considers more flexible discount rates. Shiller (1981) found that volatility of stock prices was far higher than what could be explained by the volatility of dividends and interest rates alone. This suggested that returns and price fluctuations are driven largely by discount rate premia in excess of interest rates, and that returns are to some extent predictable (Campbell & Shiller, 1988a, 1988b; Cochrane, 2008). Later this model has been applied to house prices where rents played the role of dividends (Gallin, 2008; Campbell, Davis, Gallin, & Martin, 2009; Fama & French, 2025). Changes in the price-to-rent ratio reflect mostly variations in discount rate premia and, to rule out perpetual bubbles, the ratio is assumed to be stationary.

Underpinning these models is the cointegration between house prices, rents and income. Early evidence (Malpezzi, 1999; Meen, 2002) was challenged (Gallin, 2006) but ultimately cointegration appears to hold (Gallin, 2008; Holly, Pesaran, & Yamagata, 2010). In the triangle formed by house prices, income and rent, the unit of analysis is either the price-to-income or the price-to-rent ratio. The assumption of rent-to-income stationarity is based on economic arguments.

Labor prices affect other expense categories apart from housing, particularly labor intensive services like elderly care, health care or education. Although labor prices differ across sectors, they share a common component. The Baumol effect describes how industries experiencing productivity gains drive up labor prices for other industries, even if those other industries did not experience any labor productivity at all.

This paper integrates the main features from the housing models above into a tractable lifecycle portfolio optimization framework. It considers individuals, endowed perhaps with some labor income, who plan to consume housing, labor intensive services and other products. Housing is conceptualized as a financial instrument capable of hedging not only rents, but also the labor component of remaining expense categories through the cointegration relationship between house prices, rents and income. To some extent, this framework can partly rationalize the traditional role of housing as the primary savings vehicle for conservative investors.

Creating a rich lifecycle model is only possible thanks to decades of previous academic research. Modern portfolio optimization in continuous time can be traced to Merton (1971), which provides closed form optimal consumption and investment policies for investors with constant relative risk aversion. Kim and Omberg (1996) and Wachter (2002) solved a similar problem under a stochastic but partly predictable risk premium, capturing the main features of Campbell and Shiller (1988a). Liu (2007) generalized this framework to return dynamics that follow quadratic processes, including stochastic volatility based on the Heston model or the previous stochastic risk premium model.

Housing serves a dual role in the portfolio optimization literature, as an extra type of consumption service and as an investment asset. Damgaard, Fuglsbjerg, and Munk (2003) model consumption of durable goods like houses using a Cobb-Doublas aggregator that captures complementarities with other perishable goods. Cocco (2005) considers non-tradeable labor income and Yao and Zhang (2005) also adds the possibility of renting as an alternative to homeownership, both providing housing services to be consumed but financed differently. These models are solved numerically and focus on capturing the frictions that characterize the traditional path to homeownership, such as indivisibility and illiquidity. Kraft and Munk (2011) obtain closed form solutions by removing frictions; they allow fractional investing through liquid real estate investment trusts (REITs), assume that income is tradeable and differentiate clearly between the consumption of housing services and owning a house as an investment asset. Stock, bond and housing markets are assumed to be complete enough to span income derivative contracts through a static correlation structure. Fischer and Stamos (2013) consider a similar problem to Yao and Zhang (2005) where housing risk premium is partly predictable from past returns.

Other cointegration relationships have been exploited before in lifecycle investment models. Benzoni, Collin-Dufresne, and Goldstein (2007) use the stationarity of the income-dividend ratio to predict the stock market risk premium and income growth rates, although housing is not included. Kraft, Munk, and Weiss (2019) use a Cobb-Douglas aggregator to capture the consumption of housing services and allow investing in housing markets with short-selling and leverage constraints. Sophisticated investors have two predictors at their disposal related to Shiller (1981). The corporate net payout yield is used to predict the stock and the housing risk premium. Additionally, the price-rent ratio is used as a predictor of the housing risk premium and growth rate of labor prices in a setting where labor markets are incomplete.

There is broader range of literature available, although here I can only afford to mention a few papers. Sinai and Souleles (2005) argue that homeownership can be viewed as a hedging instrument against rent prices. Kueng, Lockwood, and Pinchuan (2024) argue that young risk averse individuals may still prefer to rent because house prices are highly correlated with income and their exposure through human capital is already high. Many authors have studied how financing conditions affect house prices (Himmelberg, Mayer, & Sinai, 2005; Taylor, 2007; Duca, Muellbauer, & Murphy, 2011, 2021). Another body of the literature evaluates the role of homeownership in building household wealth (Di, Belsky, & Liu, 2007; Turner & Luea, 2009; Rappaport, 2010; Goodman & Mayer, 2018; Wainer & Zabel, 2020). These works have a sizeable relevance in public policy debates, as homeownership is considered the main savings vehicle for many households.

The main contribution of this paper is exploiting the cointegration between house prices and income to make income tradeable in a lifecycle model similar to Kraft and Munk (2011). I argue in favor of considering housing as a hedge instrument against not only rent prices but also the labor component of other consumption goods and services when focusing on long-term horizons. On the technical side, this paper extends the framework of Liu (2007) incorporating Cobb-Douglas consumption bundles and stochastic payoff stream endowments to capture consumption prices and human capital. Like in Liu (2007), exact solutions are general enough to include stochastic market price of risk or stochastic volatility components.

The structure of this paper is organized as follows. Section 1 introduces the model for consumer preferences, labor prices and financial markets, which include housing instruments. Apart from deriving house prices, it shows their implied return dynamics and describes their main features. Section 2 solves the dynamic portfolio optimization problem, describing which are the optimal policies and their associated welfare gains. For comparison, Section 3 solves a simpler static portfolio optimization problem to double check the basic results of its richer dynamic counterpart. Finally Section 4 wraps up the results and provides some concluding remarks.

1 Model

1.1 Setting

First I introduce some general processes in vector form and core assumptions that capture the generality of many results. Then I specialize these general processes to derive the main instances of a simple lifecycle investment model with housing and cointegration between rents and income. For comparison, alternative instances without cointegration are also provided. All of these notations are complementary and they are used in parallel along the paper.

General notation I assume that investors have access to investment markets and there is no arbitrage. There is a vector state process X where the drift μ_X and diffusion Σ_X may also depend themselves on X

$$\mathrm{d}X_t = \mu_X \,\mathrm{d}t + \Sigma_X \,\mathrm{d}Z_{X,t}\,.$$

The instantaneous risk-free rate is r and it may depend on state X. Cumulative risky asset returns A evolve according to dynamics

$$\frac{\mathrm{d}A_t}{A_t} = \mu_A \,\mathrm{d}t + \Sigma_A \,\mathrm{d}Z_{A,t}$$

where the drift μ_A and diffusion Σ_A may depend on state X. It is assumed that the inverse covariance matrix $(\Sigma_A \Sigma_A^\intercal)^{-1}$ exists almost surely. There exists a market price of risk Λ, which may depend on state X, satisfying the no arbitrage constraint

$$\Sigma_A \Lambda = \mu_A - r\mathbb{1} \tag{1}$$

and the pricing kernel K has dynamics

$$\frac{\mathrm{d}K_t}{K_t} = -\, r \,\mathrm{d}t - \Lambda^\intercal \,\mathrm{d}Z_{A,t}\,.$$

Investors have constant relative risk aversion (CRRA) utility with $\gamma > 0$

$$u(x) = \begin{cases} \frac{x^{1-\gamma}}{1-\gamma} & \text{if } \gamma \neq 1 \\ \log(x) & \text{if } \gamma = 1 \end{cases} \tag{2}$$

over a Cobb-Douglas consumption bundle that they maximize given some consumption budget $c \geq 0$ and prices $P > 0$

$$\begin{aligned} v(c, \theta, P) \coloneqq \max_{\xi \in \mathbb{R}_+^{|P|}} \; & (c - \xi^\intercal P)^{1-\theta^\intercal \mathbb{1}} \prod_{i=1}^{|\xi|} \xi_i^{\theta_i} \\ \text{s.t. } & \xi^\intercal P \leq c. \end{aligned} \tag{3}$$

The consumption bundle features constant returns to scale. The taste elasticities vector $\theta \geq \mathbb{0}$ over dynamically priced products satisfies $\theta^\intercal \mathbb{1} \leq 1$. The remainder of the consumption budget that was not allocated to dynamically priced products, $c - \xi^\intercal P$, is automatically allocated to a cash indexed product with a taste elasticity of $1 - \theta^\intercal \mathbb{1}$ and a unitary consumption price.

Lemma 1 (Explicit consumption bundle)**.** *The explicit solution to the consumption allocation problem* (3) *is*

$$v(c,\theta,P) = e^{(1-\theta^\intercal \mathbb{1})\log(1-\theta^\intercal \mathbb{1})+\theta^\intercal \log(\theta)-\theta^\intercal \log(P)} c \tag{4}$$

with the following allocations to dynamically priced products

$$\xi_i = \frac{\theta_i}{P_i} c$$

and to cash indexed consumption

$$c - \xi^\intercal P = (1-\theta^\intercal \mathbb{1})\, c.$$

Proof. See Section A.1 □

The dynamics of consumption prices P_t follow a stochastic process where μ_P and Σ_P may depend on state X_t

$$\frac{\mathrm{d}P_t}{P_t} = \mu_P \,\mathrm{d}t + \Sigma_P \,\mathrm{d}Z_{P,t}\,.$$

Correlation matrices between Brownian motion drivers are denoted by $\rho_{\square\square}$, for instance ρ_{XA} is the correlation matrix between Z_X and Z_A with size $|Z_X| \times |Z_A|$. Also, the duration term below helps to make expressions more concise throughout this paper

$$\mathcal{D}_{\kappa,\tau} = \int_0^\tau e^{-\kappa s}\,\mathrm{d}s = \begin{cases} \tau & \text{if } \kappa = 0 \\ \frac{1-e^{-\kappa\tau}}{\kappa} & \text{otherwise} \end{cases}.$$

Main instances The following instances implement the main mechanisms behind the lifecycle investment and housing markets models.

Financial markets have a constant risk-free rate r that investors can earn using a money market account M_t

$$\frac{\mathrm{d}M_t}{M_t} = r\,\mathrm{d}t\,.$$

Tradable risky assets A comprising a tradable stock market index S_t and instruments related to net rent price N_t, which are explained later

$$\frac{\mathrm{d}S_t}{S_t} = (r+\lambda_S\sigma_S)\,\mathrm{d}t + \sigma_S\,\mathrm{d}Z_{S,t}\,. \tag{5}$$

The stocks market price of risk λ_S is constant, and it can be interpreted as the extra returns above the risk-free rate that investors in aggregate demand for exposing their portfolio to stock market risk.

Let Y_t denote a latent income process as a geometric Brownian motion

$$\frac{\mathrm{d}Y_t}{Y_t} = \mu_Y\,\mathrm{d}t + \sigma_Y\,\mathrm{d}Z_{Y,t}\,. \tag{6}$$

The net rent price $N_t = e^{\nu_{N,t}} Y_t$ is defined in proportion to latent income Y_t through the log-wedge $\nu_{N,t}$. This log-wedge follows an Ornstein-Uhlenbeck process

$$\mathrm{d}\nu_{N,t} = \kappa_{\nu_N} \left(\bar{\nu}_N - \nu_{N,t}\right) \mathrm{d}t + \sigma_{\nu_N} \,\mathrm{d}Z_{\nu_N,t} \tag{7}$$

producing the following rent price N_t dynamics

$$\frac{\mathrm{d}N_t}{N_t} = \frac{\mathrm{d}Y_t}{Y_t} + \mathrm{d}\nu_{N,t} + \frac{1}{2}\sigma^2_{\nu_N} \,\mathrm{d}t\,. \tag{8}$$

Labor price $L_t = e^{\nu_{L,t}} Y_t$ is also defined in proportion to latent income Y_t through the log-wedge $\nu_{L,t}$, which is modelled as a Ornstein-Uhlenbeck process with dynamics

$$\mathrm{d}\nu_{L,t} = \kappa_{\nu_L} \left(\bar{\nu}_L - \nu_{L,t}\right) \mathrm{d}t + \sigma_{\nu_L} \,\mathrm{d}Z_{\nu_L,t}\,. \tag{9}$$

Net rents are traded and discounted using the constant market price of risk λ_Y for exposure to income risk $Z_{Y,t}$ and λ_{ν_N} for exposure to log-wedge risk $Z_{\nu_N,t}$. I assume that residential real estate maintenance and repair expenses accrue uniformly, and that these are already taken into account in the income-to-net-rent conversion factor $e^{\nu_{N,t}}$. In terms of the more general notation, mean reverting components $\nu_{N,t}$ and $\nu_{L,t}$ constitute the vector state process X_t. For simplicity risk factors of these simplified instances are assumed to be orthogonal to each other, unless stated otherwise.

Explicit expressions for house prices and their return dynamics are provided in Section 1.2, which are ultimately characterized by the above parameters and processes.

In this simplified setting, investors have taste elasticity θ_N to housing services with rent price N_t, and elasticity θ_L to goods and services indexed by labor price L_t, that is $P_t = (N_t, L_t)^\intercal$. This characterization is quite flexible, for instance higher housing service usage can be understood in the extensive margin as a larger house, or the intensive margin with as a house with better amenities. There is also taste elasticity $1 - \theta^\intercal \mathbb{1}$ towards constant price or cash-indexed consumption.

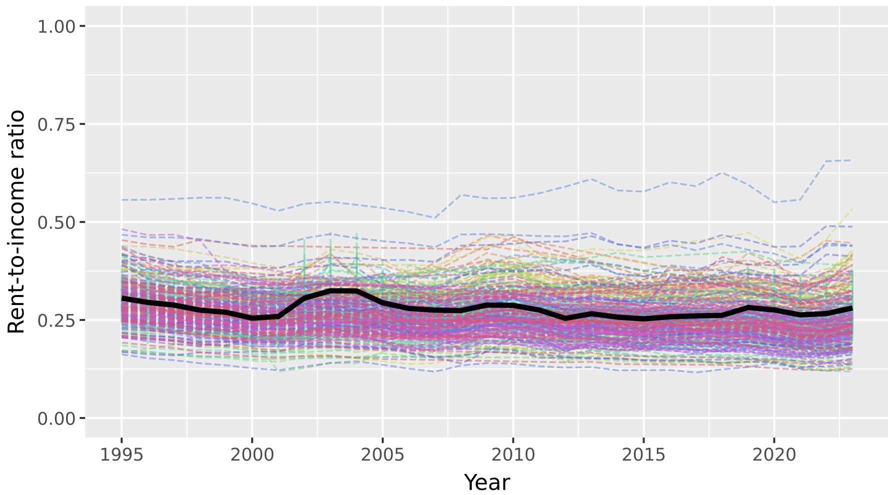
The correlation structure between rent and labor log-prices makes highly effective long-term hedging possible. The correlation implied by this model increases in the time horizon h, converging to 1 as $h \to \infty$

$$\rho_{NL,h} = \mathrm{Cor}\left(\log\left(\frac{N_{t+h}}{N_t}\right), \log\left(\frac{L_{t+h}}{L_t}\right)\right) = \frac{1}{\sqrt{1 + \frac{\sigma^2_{\nu_N}}{\sigma^2_Y}\frac{\mathcal{D}_{2\kappa_{\nu_N},h}}{h}}\sqrt{1 + \frac{\sigma^2_{\nu_L}}{\sigma^2_Y}\frac{\mathcal{D}_{2\kappa_{\nu_L},h}}{h}}}\,. \tag{10}$$

The assumption of rent-to-income stationarity is based primarily on economic arguments. Rent can be argued to have a lower bound, since providing housing services is costly, and an upper bound. It is hard to image a scenario, outside of modern society collapse, where rent reaches 100% of income. Figure 1 shows that U.S. median rents to average income ratios have been relatively stable at different large metro locations. If investors believe that fluctuations in Figure 1 around the 25% ratio are representative of future values, then approximating wedges ν_N and ν_L with constants is not farfetched.

Remark 1 (Rent-to-income simplification)**.** *This exercise motivates why fixing a constant log-wedge ν_N between rent price and income processes can be an acceptable simplification to capture long-term effects.*

Figure 1: Rent-to-income ratio at large metro locations in the U.S.



Rent data at different U.S. locations corresponds to Fair Market Rents (FMR) for a 2 bedroom unit at percentile 40 obtained from the U.S. department of Housing and Urban Development (HUD). Large metro locations are selected at the county level by their Urban Influence (IC) code obtained from the U.S. Department of Agriculture (USDA). Personal income data was obtained from the Bureau of Economic Analysis (BEA) and merged with rents at the county level. The solid black line represents a population weighted average using U.S. census data.

Suppose that log-wedges $\nu_{N,t}$ and $\nu_{L,t}$ are the mean reverting processes from (7) (9)*. One may select log-wedges parameters such that long-run labor L_t fluctuates around latent income Y_t, e.g. $e^{\bar{\nu}_L} = 1$, and net rent price N_t fluctuates around some other fraction of the latent income process Y_t, e.g. $e^{\bar{\nu}_N} = \frac{1}{5}$. Also, consider some general correlation structure with constant but non-perfect correlations $\rho_{Y\nu_N}$, $\rho_{Y\nu_L}$, $\rho_{\nu_N\nu_L}$.*

The correlation between cumulative rent price and income log-changes, using Itô isometry, is

$$\mathrm{Cor}\left(\log\left(\frac{N_{t+h}}{N_t}\right), \log\left(\frac{L_{t+h}}{L_t}\right)\right)$$
$$= \frac{1 + \frac{\sigma_{\nu_N}}{\sigma_Y}\rho_{Y\nu_N}\frac{\mathcal{D}_{\kappa_{\nu_N},h}}{h} + \frac{\sigma_{\nu_L}}{\sigma_Y}\rho_{Y\nu_L}\frac{\mathcal{D}_{\kappa_{\nu_L},h}}{h} + \frac{\sigma_{\nu_N}\sigma_{\nu_L}}{\sigma_Y^2}\rho_{\nu_N\nu_L}\frac{\mathcal{D}_{\kappa_{\nu_N}+\kappa_{\nu_L},h}}{h}}{\sqrt{1 + \frac{\sigma_{\nu_N}^2}{\sigma_Y^2}\frac{\mathcal{D}_{2\kappa_{\nu_N},h}}{h} + \frac{\sigma_{\nu_N}}{\sigma_Y}\rho_{Y\nu_N}\frac{\mathcal{D}_{\kappa_{\nu_N},h}}{h}}\sqrt{1 + \frac{\sigma_{\nu_L}^2}{\sigma_Y^2}\frac{\mathcal{D}_{2\kappa_{\nu_L},h}}{h} + \frac{\sigma_{\nu_L}}{\sigma_Y}\rho_{Y\nu_L}\frac{\mathcal{D}_{\kappa_{\nu_L},h}}{h}}}$$

We can see that as $h \to \infty$ the correlation between rent and labor prices approaches 1*. This means that we can effectively use the housing market to hedge long-term labor price changes.*

The importance of log-wedge fluctuations vanishes over the long-term. For rent prices N_t, the importance of the latent income Y_t grows linearly with horizon h

$$\log\left(\frac{N_{t+h}}{N_t}\right) = \underbrace{\left(\mu_Y - \frac{\sigma_Y^2}{2}\right)h + \sigma_Y\int_0^h \mathrm{d}Z_{Y,t+s}}_{\textit{Grows linearly with } h} + \underbrace{\nu_{N,t+h} - \nu_{N,t}}_{\textit{Stationary}}$$

while the asymptotic log-wedge distribution is stationary

$$\nu_{N,t+h} \sim \mathrm{N}\left(\bar{\nu}_N + (\nu_{N,t} - \bar{\nu}_N)e^{-\kappa_{\nu_N}h},\ \sigma_{\nu_N}^2\mathcal{D}_{2\kappa_{\nu_N},h}\right).$$

The same argument applies to labor prices L_t and labor log-wedge $\nu_{L,t}$.

Regarding financial premiums, the picture is somewhat similar. Cumulative risk premium from exposure to $Z_{Y,t}$ grows linearly with the horizon h

$$\sigma_Y \int_0^h \lambda_{Y,t+s}\,\mathrm{d}s \sim \mathrm{N}\left(\sigma_Y \bar{\lambda}_Y h + \sigma_Y(\lambda_{Y,t} - \bar{\lambda}_Y)\mathcal{D}_{\kappa_{\lambda_Y},h},\; \frac{\sigma_Y^2 \sigma_{\lambda_Y}^2}{\kappa_{\lambda_Y}^2}\left(h - \mathcal{D}_{\kappa_{\lambda_Y},h} - \frac{\kappa_{\lambda_Y}}{2}\mathcal{D}^2_{\kappa_{\lambda_Y},h}\right)\right).$$

While the instantaneous exposure of rent prices N_t to log-wedge risk $Z_{\nu_N,t}$ is constant σ_{ν_N}, the effective exposure of long-term financial instruments can be substantially lower due to the transitory mean-reverting nature of the log-wedge. Also, in as much as $Z_{\nu_N,t}$ is correlated with $Z_{Y,t}$ through $\rho_{Y\nu_N}$, exposure should command the same risk premium $\lambda_{Y,t}$ and for the remaining exposure there is a priori no reason to believe that the risk premium for $Z_{\nu_N,t}$ should be substantially larger than for $\lambda_{Y,t}$.

Thus the long-term relevance of these log-wedges is of secondary importance relative to cumulative income changes and its risk premium for many practical purposes.

Instances without cointegration To analyze the implications that cointegration has in this model, I compare results to the case in which cointegration between rents and income is assumed not to exist. These instances deviate from the main instances above in that there is no latent income process Y_t. Instead, rent and labor price are assumed to be separate geometric Brownian motion processes with possibly some correlation ρ_{NL} between innovations of $Z_{N,t}$ and $Z_{L,t}$

$$\begin{aligned}\frac{\mathrm{d}N_t}{N_t} =& \mu_N\,\mathrm{d}t + \sigma_N\,\mathrm{d}Z_{N,t}\\ \frac{\mathrm{d}L_t}{L_t} =& \mu_L\,\mathrm{d}t + \sigma_L\,\mathrm{d}Z_{L,t}\,.\end{aligned}$$

The variance of the rent and labor log-prices, as well as their correlation, are constant. This differs from the horizon dependent expression (10) in the cointegration case.

1.2 Houses, prices and returns

In this model, housing is both a consumption product and an investment asset priced according to exogenous market parameters but these two facets can be clearly disentangled. The owner of a house can freely decide to consume its housing services, to rent out the house at market prices, or to use part of the house while renting out the extra rooms. In absence of frictions, whether an individual buys a house and consumes its own housing services or whether she buys a house to rent out and uses that income to pay the rent of her main residence, are equivalent. Thus the real estate investment decision can be separated from the consumption of housing services. I also assume that fractional and frictionless investment in residential real estate is possible, for instance through a real estate investment trust (REIT).

Now it is the turn to derive the house prices and return dynamics implied by this model. Houses are understood as financial instruments whereby its owner receives a stream of ex-ante uncertain rent payments. First I derive the price for general claims to both uncertain one-time payoffs in Lemma 2 and payoff streams in Lemma 5, which includes houses. Lemma 6 shows that the price-to-rent and price-to-income ratio are stationary, in line with existing literature, under fairly general assumptions. Later I show in Lemma 7 that returns in this model are driven mainly by exposure to risk factors and their market price of risk. Afterwards I analyze the mechanics embedded inside return dynamics and some features of house prices which I compare to empirical data.

Consider the payoff process Q_t evolving according to dynamics

$$\frac{\mathrm{d}Q_t}{Q_t} = \mu_Q \,\mathrm{d}t + \Sigma_Q \,\mathrm{d}Z_{Q,t}\,. \tag{11}$$

Assumption 1. *Asset risk factors Z_A span state Z_X and payoff Z_Q risk factors*

$$\rho_{XA}\rho_{XA}^\intercal = I \qquad \rho_{QA}\rho_{QA}^\intercal = I \qquad \rho_{XA}\rho_{QA}^\intercal = \rho_{XQ}. \tag{12}$$

Lemma 2 (Terminal payoff price)**.** *The price $\Omega(t, X_t, Q_t; T)$ at time t of an uncertain payoff Q_T to be received at time T is*[1]

$$\Omega(t, X_t, Q_t; T) = Q_t \tilde{\Omega}(t, X_t; T)$$

where $\tilde{\Omega}(t, X_t; T)$ solves the following partial differential equation (PDE)

$$\begin{aligned} 0 =& \frac{\partial \tilde{\Omega}}{\partial t} + (\mu_Q - r - \Sigma_Q \rho_{QA} \Lambda)\, \tilde{\Omega}(t, X_t; T) \\ &+ \tilde{\Omega}_X^\intercal \left(\mu_X + \Sigma_X \left(\rho_{XQ}\Sigma_Q^\intercal - \rho_{XA}\Lambda\right)\right) + \frac{1}{2}\operatorname{tr}\left(\tilde{\Omega}_{XX^\intercal}\Sigma_X\Sigma_X^\intercal\right) \end{aligned} \tag{13}$$

with boundary condition $\tilde{\Omega}(T, X_T; T) = 1$. If Assumption 1 holds and Σ_A is invertible almost surely, the unique replicating strategy corresponds to

$$\pi_t = (\Sigma_A^\intercal)^{-1} \frac{\rho_{XA}^\intercal \Sigma_X^\intercal \Omega_X + \rho_{QA}^\intercal \Sigma_Q^\intercal Q_t \Omega_Q}{\Omega(t, X_t, Q_t; T)}.$$

Proof. See Section A.2 for the price and Section A.3 for the replicating strategy. □

The PDE in (13) is an instance of the generic PDE described in Lemma 3 and inherits all of its properties and solutions. Some remarks about this relationship can be found at the end of Section A.3.

Lemma 3 (Generic PDE)**.** *The following partial differential equation (PDE) with boundary condition $g(T, X_t)$ and dynamic coefficients R as a scalar, B of size $|X|$, C symmetric of size $|X| \times |X|$ and D symmetric of size $|X| \times |X|$*

$$0 = \frac{\partial g}{\partial t} + g(t, X_t)R + g_X^\intercal B + \frac{1}{2}\frac{g_X^\intercal (C - D)\, g_X}{g(t, X_t)} + \frac{1}{2}\operatorname{tr}\left(g_{XX^\intercal} D\right) \tag{14}$$

can be reduced to a system of Riccati ordinary differential equations (ODEs) under some restrictions as detailed in Section A.5, and when these Riccati equations satisfy a diagonal strucutre, they can be solved in closed form as detailed in Section A.6.

[1] Not invoking Assumption 1 to derive the price is algebraically equivalent to imputing a zero market price of risk for risk factors not spanned by Z_A. This position is debatable; if one considers the "extra" prices of risk to be undefined then Assumption 1 is required.

Lemma 4 (Generic PDE separation). *Suppose that in PDE* (14) *the state X can be partitioned into X_1 and X_2, that the boundary condition admits the following split product structure $g(T, X) = g_1(T, X_1)g_2(T, X_2)$ and that dynamic coefficients can be decomposed as follows*

$$R = R_1 + R_2 \qquad B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \qquad C = \begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix} \qquad D = \begin{pmatrix} D_1 & D_\star \\ D_\star^\intercal & D_2 \end{pmatrix} \tag{15}$$

where $D_\star$ is unrestricted but R_i, B_i, C_i and D_i for $i \in \{1, 2\}$ may depend only on state indicators X_i. Then the solution to PDE (14) *can be decomposed into*

$$g(t, X) = g_1(t, X_1)g_2(t, X_2) \tag{16}$$

where each $g_i(t, X_i)$ solves the PDE

$$0 = \frac{\partial g_i}{\partial t} + g_i(t, X_i)R_i + g_{i,X_i}^\intercal B_i + \frac{1}{2}\frac{g_{i,X_i}^\intercal (C_i - D_i)\, g_{i,X_i}}{g_i(t, X_i)} + \frac{1}{2}\operatorname{tr}\left(g_{i,X_i X_i^\intercal} D_i\right) \tag{17}$$

with boundary condition $g_i(T, X_i)$. Note that each PDE (17) *is in turn a generic PDE* (14) *as described in Lemma 3 and the separation may be applied repeatedly as long as the aforementioned conditions hold. More generally, under certain conditions detailed in Section A.7, the antidiagonal of C can be different from zero.*

Proof. See Section A.7 for the proof and the case with a non-zero C antidiagonal. □

The methodology used in Lemma 3 to reduce the generic PDE into a system of Riccati ODEs and solve them is based on Liu (2007). The diagonalization method applied in Section A.5 to the Riccati equations applies the decomposition described in Lemma 4, and in particular the extension allowing the anti-diagonals of C to be different from zero. Existence of solutions is proven only as far as those closed form solutions can reach, while uniqueness is not addressed. In general a solution may not exist, or exist only under some conditions, like a time horizon smaller than some threshold. That said, these kind of PDEs are well known in the finance literature.

The price of a house is given in Lemma 5 when using the net rent price N_t as the payoff Q_t.

Lemma 5 (Payoff stream price). *The price $\Upsilon(t, X_t, Q_t; T)$ at time t of an uncertain payoff stream Q_t with dynamics* (11) *from time t to T in terms of $\tilde{\Omega}$ from Lemma 2 is*

$$\Upsilon(t, X_t, Q_t; T) = \int_t^T \Omega(t, X_t, Q_t; s)\,\mathrm{d}s = Q_t \int_t^T \tilde{\Omega}(t, X_t; s)\,\mathrm{d}s\,. \tag{18}$$

If Assumption 1 holds and Σ_A is invertible almost surely, the unique replicating strategy corresponds invests

$$\pi_t = (\Sigma_A^\intercal)^{-1}\,\frac{\rho_{XA}^\intercal \Sigma_X^\intercal \Upsilon_X + \rho_{QA}^\intercal \Sigma_Q^\intercal Q_t \Upsilon_Q}{\Upsilon(t, X_t, Q_t; T)}$$

and distributes a continuous dividend flow of Q_t.

Proof. See Section A.8 for the price and Section A.9 for the replication strategy. □

As mentioned earlier, previous literature has found the price-to-rent and price-to-income ratios to be stationary. Lemma 6 shows that in this model those ratios are also stationary under some general assumptions.

Lemma 6 (Stationary house price to income ratio)**.** *If market parameters are Markovian and only depend on a jointly stationary state process X_t, the ratios $\Omega(t, X_t, Q_t; T)/Q_t$ and $\Upsilon(t, X_t, Q_t; T)/Q_t$ are stationary when keeping the reference time horizon $T - t$ constant or jointly stationary with X_t. Regarding house prices, the price-to-rent as well as the price-to-income ratios are also stationary.*

Proof. See Section A.10 □

It is worth to clarify that stationarity applies to reference house prices of constant (or stationary) remaining horizon like the average house price in a given region, and not to the value of a particular house. When holding the state X constant, a house that at time t is projected to last $T - t$ years will inevitably depreciate as time passes and reach a terminal value of 0 at time T. In practice the remaining horizon of a house is not fixed and owners can lengthen it by making reforms or shorten it by neglecting maintenance. In this model, we can interpret those actions as investment or divestment housing flows.

Remark 2 (Housing profitability)**.** *While house prices can effectively hedge income under the assumptions above, this does not necessarily imply that if income rises, the total return on a house will be positive. Even if income is expected to increase, expected housing total returns can remain negative.*

Consider a constant rent log-wedge ν_N, resulting in the following rent price N_t dynamics

$$\frac{\mathrm{d}N_t}{N_t} = \mu_Y \,\mathrm{d}t + \sigma_Y \,\mathrm{d}Z_{Y,t}\,. \tag{19}$$

The price of a house projected to last h years is

$$P^h_{H,t} = N_t \mathcal{D}_{r+\lambda_Y\sigma_Y-\mu_Y,h}.$$

Investing into a house produces a flow of rents and is subject to changes in the price of the house, which among other factors include depreciation. The dynamics of a self-funded portfolio W_t that continuously reinvests rents into housing reduces to

$$\frac{\mathrm{d}A^h_{H,t}}{A^h_{H,t}} = \frac{N_t}{P^h_{H,t}}\,\mathrm{d}t + \frac{\mathrm{d}P^h_{H,t}}{P^h_{H,t}} = (r + \lambda_Y\sigma_Y)\,\mathrm{d}t + \sigma_Y\,\mathrm{d}Z_{Y,t}\,.$$

The relative exposure to risk factor $Z_{Y,t}$ coincides with that of the income process Y_t in (6) *and the rent process N_t in* (19)*, meaning that this housing portfolio can completely hedge income and rent risk. The parameters of the portfolio dynamics above are static and imply an instantaneous expected return of $r + \lambda_Y\sigma_Y$. Despite income growing with a drift of μ_Y, the instantaneous expected return of this portfolio can be below the risk-free rate, or even negative, depending on the market price of rent risk λ_Y. It is possible that observed income grows because of the drift μ_Y or risk factor $Z_{Y,t}$ while realized total housing returns remain negative if $Z_{Y,t}$ is not enough to compensate a possibly negative market price of risk λ_Y.*

The observations in Remark 2 may seem sligthly counterintuitive and call for a more detailed analysis about return dynamics of investing in terminal payoff or payoff stream claims.

Cumulative returns of a terminal payoff claim $A_{\Omega,t}$ are driven solely by changes in the price of the claim, so $A_{\Omega,t} = \Omega(t, X_t, Q_t; T)$ with dynamics

$$\frac{\mathrm{d}A_{\Omega,t}}{A_{\Omega,t}} = \frac{\mathrm{d}\Omega(t, X_t, Q_t; T)}{\Omega(t, X_t, Q_t; T)}. \tag{20}$$

The compounded value of a portfolio $A_{\Upsilon,t}$ investing in a payoff stream with price $\Upsilon(t, X_t, Q_t; T)$ and reinvesting intermediate payoffs in an equivalent portfolio since time t_0 corresponds to

$$A_{\Upsilon,t} = \Upsilon(t, X_t, Q_t; T) + \int_{t_0}^{t} Q_s \frac{A_{\Upsilon,t}}{A_{\Upsilon,s}} \, \mathrm{d}s \, . \tag{21}$$

Using Itô's lemma we can arrive at the return dynamics below, that accounts for the intermediate payoff Q_t received on top of price changes in the payoff stream claim $\Upsilon(t, X_t, Q_t; T)$

$$\frac{\mathrm{d}A_{\Upsilon,t}}{A_{\Upsilon,t}} = \frac{Q_t}{\Upsilon(t, X_t, Q_t; T)} \, \mathrm{d}t + \frac{\mathrm{d}\Upsilon(t, X_t, Q_t; T)}{\Upsilon(t, X_t, Q_t; T)}. \tag{22}$$

Return dynamics become more informative when reformulated in terms of market parameters, as detailed in Lemma 7.

Lemma 7 (Payoff claims returns)**.** *The return dynamics of investing into a terminal payoff claim driven by Q_t with price $\Omega(t, X_t, Q_t; T)$ are*

$$\frac{\mathrm{d}A_{\Omega,t}}{A_{\Omega,t}} = r \, \mathrm{d}t + \Sigma_Q \left(\rho_{QA} \Lambda \, \mathrm{d}t + \mathrm{d}Z_{Q,t} \right) + \frac{\tilde{\Omega}_X^{\intercal}}{\tilde{\Omega}(t, X_t)} \Sigma_X \left(\rho_{XA} \Lambda \, \mathrm{d}t + \mathrm{d}Z_{X,t} \right) \tag{23}$$

and, if Assumption 1 holds, they collapse into

$$\frac{\mathrm{d}A_{\Omega,t}}{A_{\Omega,t}} = r \, \mathrm{d}t + \left(\rho_{QA}^{\intercal} \Sigma_Q^{\intercal} + \rho_{XA}^{\intercal} \Sigma_X^{\intercal} \frac{\tilde{\Omega}_X}{\tilde{\Omega}(t, X_t; T)} \right)^{\intercal} \left(\Lambda \, \mathrm{d}t + \mathrm{d}Z_{A,t} \right). \tag{24}$$

For a payoff stream driven by Q_t with price $\Upsilon(t, X_t, Q_t; T)$, the return dynamics are

$$\frac{\mathrm{d}A_{\Upsilon,t}}{A_{\Upsilon,t}} = r \, \mathrm{d}t + \Sigma_Q \left(\rho_{QA} \Lambda \, \mathrm{d}t + \mathrm{d}Z_{Q,t} \right) + \frac{\left(\int_t^T \frac{\partial \tilde{\Omega}(t, X_t; s)}{\partial X} \, \mathrm{d}s \right)^{\intercal} \Sigma_X}{\int_t^T \tilde{\Omega}(t, X_t; s) \, \mathrm{d}s} \left(\rho_{XA} \Lambda \, \mathrm{d}t + \mathrm{d}Z_{X,t} \right) \tag{25}$$

and, if Assumption 1 holds, they collapse into

$$\frac{\mathrm{d}A_{\Upsilon,t}}{A_{\Upsilon,t}} = r \, \mathrm{d}t + \left(\rho_{QA}^{\intercal} \Sigma_Q^{\intercal} + \rho_{XA}^{\intercal} \Sigma_X^{\intercal} \frac{\int_t^T \frac{\partial \tilde{\Omega}(t, X_t; s)}{\partial X} \, \mathrm{d}s}{\int_t^T \tilde{\Omega}(t, X_t; s) \, \mathrm{d}s} \right)^{\intercal} \left(\Lambda \, \mathrm{d}t + \mathrm{d}Z_{A,t} \right). \tag{26}$$

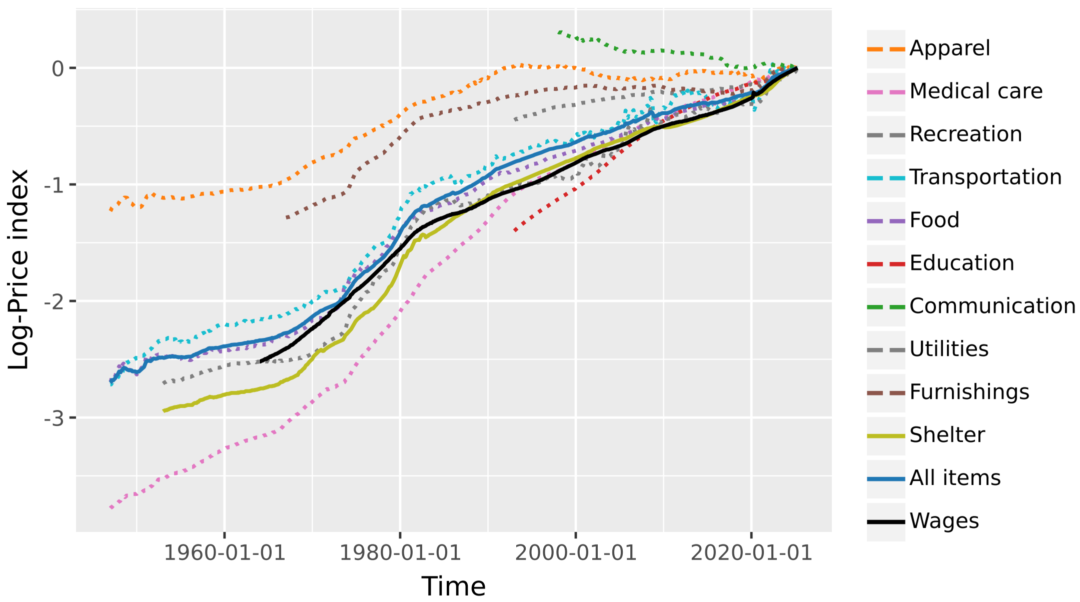
The compounded total return factor of the payoff stream claim is

$$\frac{A_{\Upsilon,t}}{A_{\Upsilon,t_0}} = \frac{\Upsilon(t, X_t, Q_t)}{\Upsilon(t_0, X_{t_0}, Q_{t_0})} e^{\int_{t_0}^{t} \frac{Q_s}{\Upsilon(s, X_s, Q_s)} \mathrm{d}s}. \tag{27}$$

Proof. See Section A.11 □

Expected returns of housing investments in this model are determined by discount rates, and more precisely, by the market price of risk Λ. In Remark 2, expected income rate μ_Y is a major driver of rents but it does not play a role in profitability since it is already incorporated

Figure 2: Wages and US Consumer Price Index



Data obtained from the Federal Reserve Bank of St. Louis. Wages refers to average hourly earnings of production and nonsupervisory employees.

into prices. The fact that price of housing services or "shelter" tracks general inflation quite well and that they have steadily increased increased, as Figure 2 shows, does not tell us the full picture. It is missing information about the risk-premium. For the purposes of individual portfolio optimization, the market price of risk λ_Y is assumed to be exogenous, but from a broader perspective it can be understood as a market equilibrium outcome. The driving factor behind this equilibrium can go both ways: it could be that investors demand some extra compensation $\lambda_Y > 0$ for bearing this risk, or that residents want to hedge rent prices and are willing to pay an insurance premium $\lambda_Y < 0$.

At this point we can analyze the implications of cointegration for housing returns in comparison to a non-cointegration model. Let us assume a constant risk-free rate r and a constant market price of risk λ_Y. In the model without cointegration, the price of future rents with an horizon of h is

$$P_{N,t}^h = N_t e^{(\mu_N - r - \lambda_N \sigma_N)h}$$

and the price of a house with the same horizon is

$$P_{H,t}^h = N_t \mathcal{D}_{r+\lambda_N\sigma_N-\mu_N,h}.$$

Return dynamics coincide for both instruments and are independent of the horizon h with constant risk exposure

$$\frac{\mathrm{d}A_{H,t}}{A_{H,t}} = \frac{\mathrm{d}A_{N,t}}{A_{N,t}} = r\,\mathrm{d}t + \sigma_N\left(\lambda_N\,\mathrm{d}t + \mathrm{d}Z_{N,t}\right). \tag{28}$$

Now let us look into the cointegration case and suppose that the mean reverting log-wedge for rent $\nu_{N,t}$ is (7). Under cointegration, house prices $P_{H,t}^h = \int_0^h P_{N,t}^s\,\mathrm{d}s$ can be described in terms of the price of future rent claims $P_{N,t}^h$ where

$$P_{N,t}^h = N_t e^{(\mu_Y - r - \lambda_Y\sigma_Y)h + \left(\kappa_{\nu_N}\left(\bar{\nu}_N - \nu_{N,t}\right) - \lambda_{\nu_N}\sigma_{\nu_N}\right)\mathcal{D}_{\kappa_{\nu_N},h} + \frac{1}{2}\sigma_{\nu_N}^2 \mathcal{D}_{2\kappa_{\nu_N},h}}.$$

For a constant horizon h, returns for rent claims are given by

$$\frac{\mathrm{d}A_{N,t}^{h}}{A_{N,t}^{h}} = r\,\mathrm{d}t + \sigma_Y\left(\lambda_Y\,\mathrm{d}t + \mathrm{d}Z_{Y,t}\right) + \sigma_{\nu_N} e^{-\kappa_{\nu_N} h}\left(\lambda_{\nu_N}\,\mathrm{d}t + \mathrm{d}Z_{\nu_N,t}\right)$$

and house investment returns are given by

$$\frac{\mathrm{d}A_{H,t}^{h}}{A_{H,t}^{h}} = r\,\mathrm{d}t + \sigma_Y\left(\lambda_Y\,\mathrm{d}t + \mathrm{d}Z_{Y,t}\right) + \sigma_{\nu_N} \frac{\int_0^h e^{-\kappa_{\nu_N} s} P_{N,t}^{s}\,\mathrm{d}s}{\int_0^h P_{N,t}^{s}\,\mathrm{d}s}\left(\lambda_{\nu_N}\,\mathrm{d}t + \mathrm{d}Z_{\nu_N,t}\right)$$

with risk exposure to $Z_{\nu_N,t}$ decreasing with the maturity h in both cases from a maximum of σ_{ν_N} at $h=0$. Intuitively, contemporary shocks to the rent-to-income ratio have a direct impact on rent payments in the near-term but the ratio is expected to mean revert over longer horizons. In the case of housing returns, exposure to $Z_{\nu_N,t}$ depends on $\nu_{N,t}$ through $\frac{\int_0^h e^{-\kappa_{\nu_N} s} P_{N,t}^{s}\mathrm{d}s}{\int_0^h P_{N,t}^{s}\mathrm{d}s} \in (0,1]$. The integrals inside this term are a bit unwieldy, so for tractability reasons I will parametrize investment decisions in terms of investment in rent claims $A_{N,t}^{h}$ and not houses $A_{H,t}^{h}$. The risk exposure of houses can be approximated by a representative rent claim through an appropriately parametrized rent claim exposure $e^{-\kappa_{\nu_N} h} \in (0,1]$. The dependence of house risk exposure on $\nu_{N,t}$ is mitigated by the time horizon acting as a stabilizer, since dependence on $\nu_{N,t}$ declines for long-dated instruments.

Incorporating a mean reverting risk-free rate r_t and market price of rent risk $\lambda_{Y,t}$ allows the model to capture richer patterns while maintaining most of the tractability. For expositional simplicity, in this case I assume that wedge $\nu_{N,t}$ is constant. Suppose that there is a mean reverting Vasicek risk-free rate r_t

$$\mathrm{d}r_t = \kappa_r(\bar{r} - r_t)\,\mathrm{d}t + \sigma_r\,\mathrm{d}Z_{r,t}$$

as well as a mean reverting market price of income risk with Ornstein-Uhlenbeck dynamics

$$\mathrm{d}\lambda_{Y,t} = \kappa_{\lambda_Y}(\bar{\lambda}_Y - \lambda_{Y,t})\,\mathrm{d}t + \sigma_{\lambda_Y}\,\mathrm{d}Z_{\lambda_Y,t}$$

where the risk factor $Z_{\lambda_Y,t}$ has itself a constant market price of risk λ_{λ_Y}. In this context, let me analyze the price of housing $P_{H,t}^{h} = \int_0^h P_{N,t}^{s}\,\mathrm{d}s$ through its integrand, the price of future rents $P_{N,t}^{h}$, since closed-form solution are available for the integrand but not for the integral. The price of a rent claim with horizon h is

$$P_{N,t}^{h} = N_t \begin{pmatrix} e^{\mu_Y h} e^{-\left(\bar{r} - \lambda_r \frac{\sigma_r}{\kappa_r} - \frac{\sigma_r^2}{2\kappa_r^2}\right)\left(h - \mathcal{D}_{\kappa_r,h}\right) - \frac{\sigma_r^2}{4\kappa_r}\mathcal{D}_{\kappa_r,h}^2 - r_t \mathcal{D}_{\kappa_r,h}} \\ \cdot\, e^{-\left(\sigma_Y \bar{\lambda}_Y - \lambda_{\lambda_Y}\frac{\sigma_Y \sigma_{\lambda_Y}}{\kappa_{\lambda_Y}} - \frac{1}{2}\frac{\sigma_Y^2 \sigma_{\lambda_Y}^2}{\kappa_{\lambda_Y}^2}\right)\left(h - \mathcal{D}_{\kappa_{\lambda_Y},h}\right) - \frac{\sigma_Y^2\sigma_{\lambda_Y}^2}{4\kappa_{\lambda_Y}}\mathcal{D}_{\kappa_{\lambda_Y},h}^2 - \lambda_{Y,t}\sigma_Y \mathcal{D}_{\kappa_{\lambda_Y},h}} \end{pmatrix} \tag{29}$$

with dynamics

$$\frac{\mathrm{d}A_{N,t}^{h}}{A_{N,t}^{h}} = r_t\,\mathrm{d}t + \sigma_Y\left(\lambda_{Y,t}\,\mathrm{d}t + \mathrm{d}Z_{Y,t}\right) - \sigma_r \mathcal{D}_{\kappa_r,h}\left(\lambda_r\,\mathrm{d}t + \mathrm{d}Z_{r,t}\right) - \sigma_{\lambda_Y}\sigma_Y \mathcal{D}_{\kappa_{\lambda_Y},h}\left(\lambda_{\lambda_Y}\,\mathrm{d}t + \mathrm{d}Z_{\lambda_Y,t}\right). \tag{30}$$

House prices above are inversely related to interest rates r_t and market price of risk $\lambda_{Y,t}$. For markets to be complete, in this case we need a bond and at least two types of house with different maturities so that their exposure to $Z_{\lambda_Y,t}$ differs. On top of the effects of income growth and interest rates effects discussed in Himmelberg et al. (2005) and Taylor (2007), the simple pricing model in (29) can capture the role of changing discount rates considered in Campbell et al. (2009) and Fama and French (2025) through the stochastic market price of risk $\lambda_{Y,t}$.

Compared to the case of a constant market price of risk where individuals can hedge rent prices by investing into any house, the introduction of a stochastic $\lambda_{Y,t}$ makes hedgers construct a housing portfolio with a remaining horizon that matches their planned consumption. That is, consumers myopically hedging income risk $Z_{Y,t}$ would incur into reinvestment risk $Z_{\lambda_Y,t}$, so a sensible risk-averse consumer should hedge both. For a person that wants to hedge rents or obtain a labor income stream, the hedge portfolio can still be a house if its remaining horizon matches the consumption plan. The residual land value of properties is absent from this model for simplicity.

Mean-reverting interest rates r_t and market price of risk $\lambda_{Y,t}$ can drive a large part of the variability in the cointegration relationship between house prices and labor income, but they are just part of a larger system. Cointegration rests markedly upon the stationary wedges $\nu_{N,t}$ and $\nu_{L,t}$ that hold rent and labor close to each other. Comparing the case of mean-reverting market price of risk $\lambda_{Y,t}$, to the one with mean reverting wedges $\nu_{N,t}$ shows that these components capture different mechanisms. From (28), exposure to wedge risk $Z_{\nu_N,t}$ diminishes in remaining horizon h

$$\sigma_{\nu_N} e^{-\kappa_{\nu_N} h} = \begin{cases} \sigma_{\nu_N} & \text{if } h = 0 \\ 0 & \text{if } h \to \infty \end{cases}$$

while from (30) we can see that absolute exposure to changes in market price of risk grows in remaining horizon h

$$-\mathcal{D}_{\kappa_{\lambda_Y},h}\sigma_{\lambda_Y}\sigma_Y = \begin{cases} 0 & \text{if } h = 0 \\ -\frac{1}{\kappa_{\lambda_Y}}\sigma_{\lambda_Y}\sigma_Y & \text{if } h \to \infty \end{cases}.$$

For expositional clarity I focus mainly on the wedges $\nu_{N,t}$ and $\nu_{L,t}$, but the general model is much richer thanks to additional mechanisms embedded into the state process X_t, at the cost of some extra complexity.

Previously I made some simplifying assumptions and design choices that shaped the resulting model for house prices. To motivate that they are to some extent plausible, I analyze house transaction data of King county in Washington state from 2014-2015 by Center for Spatial Data Science (2020) and review the findings of Holly et al. (2010). Apart from showing that data is compatible with the house price model, I also want to explain how to reconcile some apparent differences. Column (1) from Table 1 shows that the elasticity between house prices and local earnings is close to 1 as implied by the model, yet this observation may appear more comforting that it actually is. King county fits in a diameter of about 75 miles and 99.9% of home sales are located in 48 miles diameter, making most locations within commuting distance. At this micro scale, cross-sectional differences in prices can reveal more about differences in house characteristics than about location-specific labor opportunities, as column (2) suggests. For the elasticity question I think that Holly et al. (2010) provide a better analysis; they estimate the house price to income elasticity applying more appropriate methodology to a panel data of US states. They use common correlated effects to exploit time series variation in income and house prices at sufficiently separate locations, while controlling for unobserved common factors. They estimate an elasticity of house prices to income slightly higher than unity (1.14-1.2) and

Table 1: Home Sales in King County, WA (2014-2015)

	Dependent variable: log(house price)		
	(1)	(2)	(3)
Intercept	1.344***	2.995***	8.034***
	(0.092)	(0.125)	(0.097)
log(local earnings)	1.038***	0.638***	
	(0.008)	(0.006)	
log(living area sq. feet)		0.390***	0.463***
		(0.007)	(0.005)
log(extra lot sq. feet)		-0.058***	0.056***
		(0.002)	(0.002)
grade[6.Lowest in code]		0.119***	0.055**
		(0.025)	(0.022)
grade[7.Average]		0.186***	0.130***
		(0.024)	(0.022)
grade[8.Above average]		0.268***	0.236***
		(0.025)	(0.022)
grade[9.Better architect.]		0.425***	0.378***
		(0.026)	(0.022)
grade[10.High quality]		0.573***	0.485***
		(0.027)	(0.023)
grade[11+.Custom design]		0.791***	0.654***
		(0.030)	(0.025)
condition[2.Fair-badly worn]		0.000	0.122
		(0.099)	(0.092)
condition[3.Average]		0.055	0.244***
		(0.093)	(0.088)
condition[4.Good]		0.115	0.272***
		(0.093)	(0.088)
condition[5.Very good]		0.238**	0.342***
		(0.093)	(0.088)
trend & season	no	yes	yes
views	no	yes	yes
zipcode	no	no	yes
Observations	21409	21401	21427
Adjusted R^2	0.460	0.727	0.882
F Statistic	16196.788***	2490.849***	1630.380***
Note:			*p<0.1; **p<0.05; ***p<0.01

Local earnings refers to mean earnings in 2014 for the U.S. census tract where the house is located. The variable "extra lot sq. feet" is computed as the lot area minus the average living area per floor. The intercept corresponds to a low quality building grade 1-5 with a poor- worn out condition. Not part of the above regressions, but coefficients are robust to controls for bedrooms and bathrooms. Standard Errors are heteroscedasticity robust (HC3).

argue that an elasticity of 1 cannot be rejected. The power of this test is not reported so this position relies partly on the economic arguments behind the model.

The interesting part about Table 1 comes from interpreting building grade and condition as proxies for durability q in the house price model. For this purpose I prefer the hedonic regression of column (3) from Table 1 instead of column (2). While there may be reasons for people to pay a hefty price to live near high income earners, e.g. pursuing social status or as a proxy for lower criminality, more likely it captures some unobserved house characteristics that people with higher income are willing to pay for, like local amenities. It may be more informative then to use location fixed effects instead of controlling just for local earnings, as in column (3). In any case, we can observe that higher building grade and condition are positively and strongly linked to higher house prices. I cannot rule out that home buyers prefer well maintained houses with higher quality materials and architectural design for reasons other than durability, but this could still be in line with the model. As long as these durable attributes allow property owners to capture rents for longer time horizons or higher rents for comparable horizons, higher house prices can be justified.

2 Dynamic portfolio optimization

Suppose that an individual with initial wealth W_t at time t seeks to maximize expected CRRA utility (2) until time T over a Cobb-Douglas consumption bundle (4). This bundle captures complementarities between different products, like housing services or recreation, and is more uncertainty since prices are stochastic. Utility is derived from an instantaneous consumption flow weighted by ε_1, and consumption of terminal wealth weighted by ε_2. Both types of consumption are discounted with impatience rate δ, which may depend on the state process X_t.

$$J(t, W_t, X_t, P_t, Q_t) = \sup_{\pi, c} E\left[\varepsilon_1 \int_t^T e^{-\int_t^s \delta_q \mathrm{d}q} u(v(c_s, \tilde{\theta}, P_s))\, \mathrm{d}s + \varepsilon_2 e^{-\int_t^T \delta_q \mathrm{d}q} u(v(W_T, \theta, P_T))\right] \tag{31}$$

$$\text{s.t. } \frac{\mathrm{d}W_t}{W_t} = \frac{Q_t \mathbb{1}_{t \leq T_R} - c_t}{W_t}\, \mathrm{d}t + (\pi^\intercal(\mu_A - r\mathbb{1}) + r)\, \mathrm{d}t + \pi^\intercal \Sigma_A\, \mathrm{d}Z_{A,t} \tag{32}$$

The investor controls the fraction of wealth π_t to invest into assets A and the consumption at every instant c_t. Wealth dynamics reflect the stochastic nature of investments describe in Section 1.1 and the portfolio is self-financed except if $Q_t \neq 0$ with positive probability. The process Q_t captures a stochastic endowment stream, like labor income, lasting until time $T_R \leq T$ with exogenous dynamics (11). Elasticity of intermediate and terminal consumption are given by $\tilde{\theta}$ and θ respectively. Prices of consumption goods and services are modelled through P_t and indirect utility is denoted by J.

Proposition 1 (Dynamic portfolio optimization)**.** *The optimal consumption and investment* *are given by*

$$
\begin{aligned}
c_t^\star =&\varepsilon_1^{\frac{1}{\gamma}} \left(e^{(1-\tilde{\theta}^\intercal \mathbb{1}) \log\left(1-\tilde{\theta}^\intercal \mathbb{1}\right)+\tilde{\theta}^\intercal \log\left(\tilde{\theta}\right)-\tilde{\theta}^\intercal \log(P_t)}\right)^{\frac{1}{\gamma}-1} J_W^{-\frac{1}{\gamma}} && (33)\\
\pi_t^\star =& \left(\Sigma_A \Sigma_A^\intercal\right)^{-1} \left(\mu_A - r\mathbb{1}\right) \frac{J_W}{-J_{WW} W_t}\\
&+ \left(\Sigma_A \Sigma_A^\intercal\right)^{-1} \Sigma_A \rho_{XA}^\intercal \Sigma_X^\intercal \frac{J_{WX}}{-J_{WW} W_t}\\
&+ \left(\Sigma_A \Sigma_A^\intercal\right)^{-1} \Sigma_A \rho_{PA}^\intercal \Sigma_P^\intercal \operatorname{diag}(P_t) \frac{J_{WP}}{-J_{WW} W_t}\\
&+ \left(\Sigma_A \Sigma_A^\intercal\right)^{-1} \Sigma_A \rho_{QA}^\intercal \Sigma_Q^\intercal Q_t \frac{J_{WQ}}{-J_{WW} W_t}. && (34)
\end{aligned}
$$

Indirect utility is the solution to the PDE (A.28)*.*

Proof. See Section A.12 for the proof and the indirect utility PDE. □

The solution proposed in Proposition 1 is based on the dynamic programming principle. To apply this method I implicitly assume that indirect utility satisfies some differentiability conditions. As in Lemma 3, existence and uniqueness of solutions to the PDE above are not addressed. Thus the solutions obtained through dynamic programming in this paper may be regarded formally as candidate solutions.

The optimal investment fraction (34) from Proposition 1 has four distinct components: speculative demand maximizing the expected return in relation to risk exposure on the first line, hedging demand against changes in state X_t on the second line, hedging demand against changes in consumption prices P_t on the third line and hedging against changes in endowment payoffs Q_t on the fourth line. The structure of the hedging demands resembles that of slope coefficients in ordinary least squares (OLS), as these terms include the covariance between assets A and the variable of interest divided by the variance of assets A.

It is helpful to parametrize A in a way such that the covariance matrix $\Sigma_A \Sigma_A^\intercal$ is easy to invert and expressions are easy to interpret. One can parametrize controls in terms of exposure to isolated risk factors to diagonalize the covariance matrix, and then translate the optimal investment fraction in terms of the original investment assets. For the housing model, one can first solve for the optimal exposure to risk factors, e.g. $Z_{N,t}$, and then reparametrize in terms of housing allocation by matching the sensitivity of house prices to the risk factors through numerical methods.

Optimal instantaneous consumption (33) decreases in marginal utility of savings J_W and increases in marginal utility of consumption. The net effect of prices is unclear at this stage since marginal utility of saving J_W is also affected by prices and could cancel out.

As in Liu (2007), explicit solutions generally assume that markets are complete, except when restricted to terminal wealth problems which can be solved explicitly even if markets are incomplete.

Proposition 2 (Dynamic portfolio optimization with incomplete markets)**.** *Assuming that the investor is not endowed with any payoff claim and removing intermediate consumption*

$$
Q_t = 0 \qquad \varepsilon_1 = 0 \qquad \varepsilon_2 = 1, \tag{35}
$$

the optimal investment fraction is

$$\begin{aligned}\pi_t^\star =&(\Sigma_A\Sigma_A^\intercal)^{-1}\frac{\mu_A-r\mathbb{1}}{\gamma}\\&+(\Sigma_A\Sigma_A^\intercal)^{-1}\Sigma_A\rho_{XA}^\intercal\Sigma_X^\intercal\frac{h_X}{h(t,X_t)}\\&+\left(1-\frac{1}{\gamma}\right)(\Sigma_A\Sigma_A^\intercal)^{-1}\Sigma_A\rho_{PA}^\intercal\Sigma_P^\intercal\theta.\end{aligned}\tag{36}$$

Indirect utility is given by

$$J(t,W_t,X_t,P_t)=\frac{W_t^{1-\gamma}}{1-\gamma}f(t,X_t,P_t)^\gamma\tag{37}$$

where

$$f(t,X_t,P_t)=\left(e^{(1-\theta^\intercal\mathbb{1})\log(1-\theta^\intercal\mathbb{1})+\theta^\intercal\log(\theta)-\theta^\intercal\log(P_t)}\right)^{\frac{1}{\gamma}-1}h(t,X_t)$$

and $h(t,X_t)$ solves the following PDE

$$\begin{aligned}\frac{\partial h}{\partial t}=&h(t,X_t)\left(\frac{\delta}{\gamma}-\left(\frac{1}{\gamma}-1\right)\left(\begin{aligned}&r+\frac{1}{2}\frac{(\mu_A-r\mathbb{1})^\intercal(\Sigma_A\Sigma_A^\intercal)^{-1}(\mu_A-r\mathbb{1})}{\gamma}\\&-\theta^\intercal\left(\mu_P+\left(\frac{1}{\gamma}-1\right)\Sigma_P\rho_{PA}\Sigma_A^\intercal(\Sigma_A\Sigma_A^\intercal)^{-1}(\mu_A-r\mathbb{1})\right)\\&+\frac{1}{2}\left(\frac{1}{\gamma}-1\right)\theta^\intercal\Sigma_P\left(\gamma I+(1-\gamma)\rho_{PA}\rho_{PA}^\intercal\right)\Sigma_P^\intercal\theta\\&+\frac{1}{2}\operatorname{tr}\left(\operatorname{diag}(\theta)\Sigma_P\Sigma_P^\intercal\right)\end{aligned}\right)\right)\\&-h_X^\intercal\left(\mu_X+\left(\frac{1}{\gamma}-1\right)\Sigma_X\left(\rho_{XA}\Sigma_A^\intercal(\Sigma_A\Sigma_A^\intercal)^{-1}(\mu_A-r\mathbb{1})-(\gamma\rho_{XP}+(1-\gamma)\rho_{XA}\rho_{PA}^\intercal)\Sigma_P^\intercal\theta\right)\right)\\&-\frac{1}{2}(1-\gamma)h_X^\intercal\Sigma_X\left(\rho_{XA}\rho_{XA}^\intercal-I\right)\Sigma_X^\intercal h_X\frac{1}{h(t,X_t)}-\frac{1}{2}\operatorname{tr}\left(h_{XX^\intercal}\Sigma_X\Sigma_X^\intercal\right)\end{aligned}\tag{38}$$

with boundary condition $h(T,X_t)=1$.

Proof. See Section A.13 ☐

Regarding technical aspects, the PDE in (38) is an instance of the generic PDE described in (3) and inherits all of its properties and solutions. Some remarks about this relationship can be found at the end of Section A.13. Solutions cover the cases of stochastic price of risk and stochastic volatility as in Liu (2007). The dynamics of $h(t,X_t)$ share the same structure as the multiplier $\tilde{\Omega}(t,X_t)$ used to capitalize a future payoff Q_t at market value (Lemma 2). Taking derivatives on both sides of (37) with respect to wealth W_t and rearraging terms,

$$J_W=\left.\frac{\partial u(v(c,\theta,P_t))}{\partial c}\right|_{c=\frac{W_t}{h(t,X_t)}}$$

one can interpret $h(t,X_t)$ as the subjective value of money that is invested according to optimal policy up to time T in comparison to money that would have to be spent now, capturing the potential of making profits by the investment policy as well as distaste for risk, impatience and any hedging related sacrifice. Moreover when $\gamma\to\infty$, $P_t=Q_t$, $\theta=1$ and Assumption 1 holds, the dynamics of $h(t, X_t)$ match those of $\tilde{\Omega}(t, X_t)$ with an imputed price of risk $\Lambda = \Sigma_A^\intercal (\Sigma_A \Sigma_A^\intercal)^{-1} (\mu_A - r\mathbb{1})$, making $f(t, X_t, P_t)$ equal to the market price of a future consumption unit $\Omega(t, X_t, P_t; T)$. If instead $\theta = 0$, then $f(t, X_t, 1) = h(t, X_t) = \Omega(t, X_t, 1; T)$ match the price of a zero coupon bond at the previously imputed price of risk.

The optimal policy in (36) has an speculative component on the first line that is directly proportional to the market price of risk Λ and inversely proportional to risk aversion and volatility. Hedging demand contains terms resembling OLS slope coefficients, and the component hedging against consumption prices simply multiplies the OLS slope by $1 - \frac{1}{\gamma}$ and the corresponding elasticity θ. Despite hedging against consumption prices P_t, the investment policy does not depend on price levels.

Let us analyze how the optimal investment strategy of Proposition 2 with incomplete markets differs in the case of cointegration from the non-cointegration case. Suppose that labor markets are incomplete and both the risk-free rate r and market price of risk λ_Y are constant. The optimal investment strategy in the absence of cointegration is

$$\pi^\star_{N,t} = \frac{1}{\gamma}\frac{\lambda_N}{\sigma_N} + \left(1 - \frac{1}{\gamma}\right)\left(\theta_N + \rho_{NL}\frac{\sigma_L}{\sigma_N}\theta_L\right) \tag{39}$$

Assuming cointegration with complete housing market, the optimal investment strategy is

$$\pi^\star_{Y,t} = \frac{1}{\gamma}\frac{\lambda_Y}{\sigma_Y} + \left(1 - \frac{1}{\gamma}\right)(\theta_N + \theta_L) \tag{40}$$

$$\pi^\star_{\nu_N,t} = e^{\kappa_{\nu_N} h}\left(\frac{1}{\gamma}\frac{\lambda_{\nu_N}}{\sigma_{\nu_N}} + \left(1 - \frac{1}{\gamma}\right)e^{-\kappa_{\nu_N}(T-t)}\theta_N\right) \tag{41}$$

In the case above, the investment strategy is formulated for assets that differ from those available in the non-cointegration case (39). The comparison is simpler when housing markets are incomplete, then the optimal investment strategy under cointegration is

$$\pi^\star_{N,t} = \frac{1}{\gamma}\frac{\lambda_Y\sigma_Y + \lambda_{\nu_N}\sigma_{\nu_N}e^{-\kappa_{\nu_N}h}}{\sigma_Y^2 + \sigma_{\nu_N}^2 e^{-2\kappa_{\nu_N}h}} + \left(1 - \frac{1}{\gamma}\right)\left(\frac{\sigma_Y^2 + \sigma_{\nu_N}^2 e^{-\kappa_{\nu_N}(T-t+h)}}{\sigma_Y^2 + \sigma_{\nu_N}^2 e^{-2\kappa_{\nu_N}h}}\theta_N + \frac{\sigma_Y^2}{\sigma_Y^2 + \sigma_{\nu_N}^2 e^{-2\kappa_{\nu_N}h}}\theta_L\right) \tag{42}$$

All models share the same optimal policy for stocks, known as the Merton fraction

$$\pi^\star_{S,t} = \frac{1}{\gamma}\frac{\lambda_S}{\sigma_S}, \tag{43}$$

however they showed quite some disagreement about the housing market. Differences in speculative terms for the housing market are driven by the avilability of different instruments or by the diversification possibilities of the orthogonal dependence structure. Hedging demands consist in the product of 3 terms: $1 - \frac{1}{\gamma}$, the OLS slope of each consumption price with respect to the investment instrument and the respective product elasticity θ. In case that wedge mean reversion is strong or the rent instrument has long maturity $\kappa_{\nu_N} h \to \infty$, the optimal investment fraction into the rent instrument $\pi^\star_N$ under incomplete housing markets (42) tends towards the optimal investment fraction into latent income $\pi^\star_Y$ under complete housing markets (40). The most remarkable difference with respect to non-cointegration is that hedging positions for labor prices are directly proportional to θ_L in (40) whereas in absence of cointegration, it reduces to $\rho_{NL}\theta_L$ in (39) supposing that $\sigma_N \gtrapprox \sigma_L$. Thus cointegration between house and labor prices makes long-term hedging of labor much more effective.

Table 2: Model parameters

Main parameters (cointegration)

$T-t=40$	$\delta=0.01$	$r=0.03$	$\sigma_S=0.2$
$\lambda_S=0.07/0.2$	$h=50$	$\theta_N=0.2$	$\theta_L=0.4$
$\mu_Y=0.04$	$\sigma_Y=0.08$	$\lambda_Y=0.02/0.08$	$\bar{\nu}_N=\log(0.2)$
$\kappa_{\nu_N}=0.3$	$\sigma_{\nu_N}=0.06$	$\lambda_{\nu_N}=0$	$\bar{\nu}_L=0$
$\kappa_{\nu_L}=0.3$	$\sigma_{\nu_L}=0.08$		

Implied correlation structure under cointegration

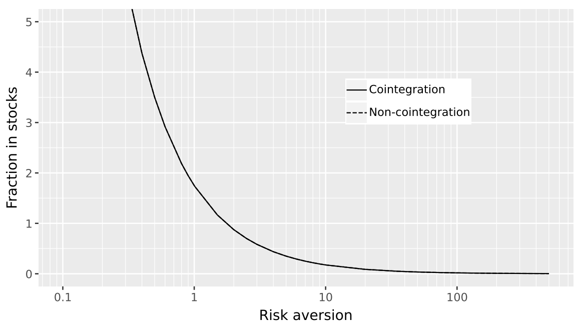
$\rho_{NL,1}=0.6333$	$\rho_{NL,10}=0.8855$	$\rho_{NL,50}=0.9746$

Non-cointegration parameters

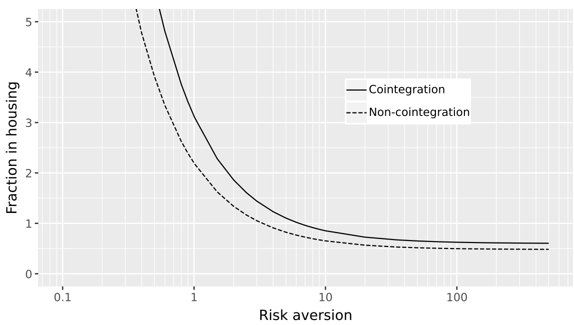
$\rho_{NL}=0.6333$	$\sigma_L=0.1059$	$\sigma_N=0.0954$	$\lambda_N=0.02/0.0954$

For the non-cointegration scenario the parameters correspond to precise estimates of 1 year statistics. The state is evaluated at its stationary values.

Figure 3: Investment strategies, incomplete markets



(a) Fraction in stocks



(b) Fraction in housing

Parameters are described in Table 2.

Figure 3 shows graphically how the investment strategies change for different investors according to their risk aversion. The comparison between cointegration and non-cointegration scenarios takes place using the parameters describe in Table 2.

To solve the portfolio optimization problem with intermediate consumption and endowment stream Q_t, I assume that markets are complete. Allowing for intermediate consumption and endowment stream Q_t makes it possible to capture much richer mechanisms, however analytical solutions are obtained only if markets are assumed to be complete.

Assumption 2. *All asset risk factors are tradeable, that is, the diffusion matrix Σ_A is invertible. Asset risk factors Z_A span state X and price P risk factors*

$$\rho_{XA}\rho_{XA}^\intercal = I \qquad \rho_{PA}\rho_{PA}^\intercal = I \qquad \rho_{XA}\rho_{PA}^\intercal = \rho_{XP} \tag{44}$$

and, if $P[Q_t \neq 0] > 0$, they also span payoff Q risk factors with

$$\rho_{QA}\rho_{QA}^\intercal = I \qquad \rho_{XA}\rho_{QA}^\intercal = \rho_{XQ} \qquad \rho_{PA}\rho_{QA}^\intercal = \rho_{PQ}.$$

Proposition 3 (Dynamic portfolio optimization with complete markets)**.** *Under Assumption 2, the optimal consumption and investment are given by*

$$c_t^\star = \varepsilon_1^{\frac{1}{\gamma}} \left(e^{(1-\tilde{\theta}^\intercal \mathbb{1})\log\left(1-\tilde{\theta}^\intercal\mathbb{1}\right)+\tilde{\theta}^\intercal \log\left(\tilde{\theta}\right)-\tilde{\theta}^\intercal\log(P_t)}\right)^{\frac{1}{\gamma}-1} f(t,X_t,P_t)^{-1}\left(W_t + \Upsilon(t,X_t,Q_t;T_R)\right) \tag{45}$$

$$\begin{aligned}
\pi_t^\star =& \left(\Sigma_A\Sigma_A^\intercal\right)^{-1}\frac{(\mu_A - r\mathbb{1})}{\gamma}\frac{W_t+\Upsilon(t,X_t,Q_t;T_R)}{W_t} \\
&+ \left(\Sigma_A\Sigma_A^\intercal\right)^{-1}\Sigma_A\rho_{XA}^\intercal\Sigma_X^\intercal\left(f(t,X_t,P_t)^{-1}f_X\frac{W_t+\Upsilon(t,X_t,Q_t;T_R)}{W_t} - \frac{\Upsilon_X}{W_t}\right) \\
&+ \left(\Sigma_A\Sigma_A^\intercal\right)^{-1}\Sigma_A\rho_{PA}^\intercal\Sigma_P^\intercal \operatorname{diag}(P_t) f(t,X_t,P_t)^{-1} f_P \frac{W_t+\Upsilon(t,X_t,Q_t;T_R)}{W_t} \\
&- \left(\Sigma_A\Sigma_A^\intercal\right)^{-1}\Sigma_A\rho_{QA}^\intercal\Sigma_Q^\intercal\frac{\Upsilon(t,X_t,Q_t;T_R)}{W_t}
\end{aligned} \tag{46}$$

where

$$\begin{aligned}
f(t,X_t,P_t) =& \varepsilon_2^{\frac{1}{\gamma}}\left(e^{(1-\theta^\intercal\mathbb{1})\log(1-\theta^\intercal\mathbb{1})+\theta^\intercal\log(\theta)-\theta^\intercal\log(P_t)}\right)^{\frac{1}{\gamma}-1} h(t,X_t;T,\theta) \\
&+ \varepsilon_1^{\frac{1}{\gamma}}\left(e^{(1-\tilde{\theta}^\intercal\mathbb{1})\log\left(1-\tilde{\theta}^\intercal\mathbb{1}\right)+\tilde{\theta}^\intercal\log\left(\tilde{\theta}\right)-\tilde{\theta}^\intercal\log(P_t)}\right)^{\frac{1}{\gamma}-1}\int_t^T h(t,X_t;s,\tilde{\theta})\,\mathrm{d}s
\end{aligned}$$

$h(t,X_t;T,\theta)$ is a particular case of $h(t,X_t)$ described in (38) *under the restrictions of Assumption 2 with parametrized terminal date T and consumption elasticity θ; and $\Upsilon(t,X_t,Q_t;T_R)$ corresponds to the price from Lemma 5. Indirect utility is given by*

$$J(t,W_t,X_t,P_t) = \frac{\left(W_t+\Upsilon(t,X_t,Q_t;T_R)\right)^{1-\gamma}}{1-\gamma} f(t,X_t,P_t)^\gamma.$$

Proof. See Section A.14 □

On the technical side, $h(t,X_t;T,\theta)$ inherits the reduction to Riccati ODEs and the closed form solutions mentioned in Lemma 3. Solutions cover the cases of stochastic price of risk and stochastic volatility as in Liu (2007). Given the similarities in dynamics, the term $f(t,X_t,P_t)$

aggregates over all future consumption bundles in a similar manner as $\Upsilon(t, X_t, Q_t; T_R)$ capitalized the payoff stream Q_t in Lemma 5. The human capital PDE in Proposition 3 coincides with the price described in Lemma 5 and inherits its closed form solutions.

Optimal consumption (45) in Proposition 3 increases by an amount proportional to the capitalized payoff stream $\Upsilon(t, X_t, Q_t; T_R)$ when receiving a payoff stream $Q_t \neq 0$. The optimal investment fraction (46) also incorporates the extra wealth afforded by the capitalized payoff stream $\Upsilon(t, X_t, Q_t; T_R)$ but compensates for the builtin exposure to the state vector Υ_X on the second line and to tradeable risk factors through $\rho_{QA}^\intercal \Sigma_Q^\intercal$ on the last line. When intermediate and terminal elasticities coincide $\tilde{\theta} = \theta$, the optimal investment fraction and instantaneous consumption do not depend on price levels P_t.

Regarding the cointegration model, the investment policy takes advantage of both the high rent and labor hedging effectiveness afforded by housing instruments as well as human capital. Assume that elasticites coincide $\tilde{\theta} = \theta$ and, as motivated in Remark 1, wedges ν_N, ν_L are constant making the housing market complete. Then the optimal investment is a leveraged version of (40) adjusted to compensate builtin endowment exposure

$$\pi^\star_{S,t} = \frac{\lambda_S}{\gamma \sigma_S}\left(1 + \frac{Y_t \mathcal{D}_{r+\sigma_Y\lambda_Y-\mu_Y, \max(T_R-t,0)}}{W_t}\right) \tag{47}$$

$$\pi^\star_{N,t} = \left(\frac{\lambda_Y}{\gamma\sigma_Y} + \left(1 - \frac{1}{\gamma}\right)\theta\right)\left(1 + \frac{Y_t \mathcal{D}_{r+\sigma_Y\lambda_Y-\mu_Y, \max(T_R-t,0)}}{W_t}\right) - \frac{Y_t \mathcal{D}_{r+\sigma_Y\lambda_Y-\mu_Y, \max(T_R-t,0)}}{W_t}, \tag{48}$$

and intermediate consumption is

$$c^\star_t = \frac{\varepsilon_1^{\frac{1}{\gamma}}}{\varepsilon_2^{\frac{1}{\gamma}} h(t; T, \theta) + \varepsilon_1^{\frac{1}{\gamma}} \int_t^T h(t; s, \theta)\, \mathrm{d}s}\left(W_t + Y_t \mathcal{D}_{r+\sigma_Y\lambda_Y-\mu_Y, \max(T_R-t,0)}\right).$$

The housing investment fraction (48) captures two important mechanisms mentioned in the literature. As argued by Sinai and Souleles (2005), housing is used to hedge rents in proportion to housing consumption elasticity θ. Kueng et al. (2024) argue that young risk averse individuals may prefer to rent a house instead of buying because house prices are highly correlated with income and their exposure through human capital is already high. The last term in (48) takes into account the builtin exposure of human capital to income risk and subtracts it from the housing investment.

Remark 3 (Lifecycle under extreme risk aversion). *The intermediate optimal bundles consumption rate $v(c^\star_t, \tilde{\theta}, P_t)$ becomes constant as $\gamma \to \infty$*

$$v(c^\star_t, \tilde{\theta}, P_t) = \frac{W_t + \Upsilon(t, X_t, Q_t; T_R)}{f(t, X_t, P_t)} = \frac{W_0 + \Upsilon(0, X_0, Q_0; T_R)}{f(0, X_0, P_0)}. \tag{49}$$

Furthermore, if intermediate and terminal product elasticities coincide $\tilde{\theta} = \theta$ and the optimal bundle price $P_{v,t} = \left(e^{(1-\theta^\intercal \mathbb{1})\log(1-\theta^\intercal \mathbb{1}) + \theta^\intercal \log(\theta) - \theta^\intercal \log(P_t)}\right)^{-1}$ from Lemma 1 is related to the payoff process Q_t through a function that only depends on state X_t, i.e. $\frac{P_{v,t}}{Q_t} = e^{\nu_v(X_t)}$, then wealth-to-consumption-bundle-price $\frac{W_t}{P_{v,t}}$, wealth-to-payoff $\frac{W_t}{Q_t}$, wealth-to-bundle-plan-price $\frac{W_t}{f(t,X_t,P_t)}$ and wealth-to-capitalized-payoff-stream $\frac{W_t}{\Upsilon(t,X_t,Q_t;T_R)}$ ratios

i. depend on state X_t but not on payoff Q_t or price P_t levels *ii.* *are stationary when parameters are Markovian, parameters only depend on a jointly stationary state process X_t and the time horizons $T-t$, T_R-t are kept constant*

iii. *are deterministic processes when there is no state X_t*

iv. *are constant when the proportion $\frac{P_{v,t}}{Q_t} = e^{\bar{\nu}_v}$ is constant, there is only intermediate consumption with the payoff stream lasting for the whole time horizon $\varepsilon_1 = 1, \varepsilon_2 = 0, T_R = T$ and, for $\frac{W_t}{P_{v,t}}$ and $\frac{W_t}{Q_t}$, there is also no initial wealth $W_0 = 0$.*

Additionally, when the proportion $\frac{P_{v,t}}{Q_t} = e^{\bar{\nu}_v}$ is constant and there is only intermediate consumption $\varepsilon_1 = 1, \varepsilon_2 = 0$, the bundle plan value $f(t, X_t, P_t)$ becomes a constant proportion of the annuity payoff stream for Q_t at market value with a time horizon T, $f(t, X_t, P_t) = e^{\bar{\nu}_v}\Upsilon(t, X_t, Q_t; T)$.

Proof. See Section A.15. □

Lifecycle paths simplify considerably under Remark 3 when risk aversion is extreme $\gamma \to \infty$, the payoff process and the bundle price are cointegrated and product elasticities of intermediate and terminal consumption coincide $\tilde{\theta} = \theta$. For these investors, the bundle price $P_{v,t}$ and payoff Q_t become natural numéraires. That makes it convenient to express wealth in years of salary saved and consumption rates in consumption bundles per year. Stationarity in (ii.) must be understood for a representative individual at some fixed point in a reference lifecycle that started with some given initial wealth ratio. These simplifications stem from a constant bundle consumption rate (49), which coincides with the number of lifetime consumption plans that financial wealth W_t and human capital $\Upsilon(t, X_t, Q_t; T_R)$ can buy. Although Proposition 3 covers mechanisms such as stochastic market price of risk or stochastic volatility, extremely risk averse investors rely on market completeness to hedge them in Equation 46. In relation to housing markets, these investors carefully choose the duration of their housing portfolios to math their consumption needs and avoid the reinvestment risk associated to housing cycles. An example would be an apartment leasehold with the same duration as the remaining lifetime of the leaseholder, or a house that is expected to need major renovations just after that date.

For the cointegration model with complete markets, these simplifications imply that people would only invest in housing

$$\pi^{\star}_{S,t} = 0 \qquad \pi^{\star}_{N,t} = 1$$

and the number of consumed bundles reduces to the number of annuities that financial wealth and human capital can buy

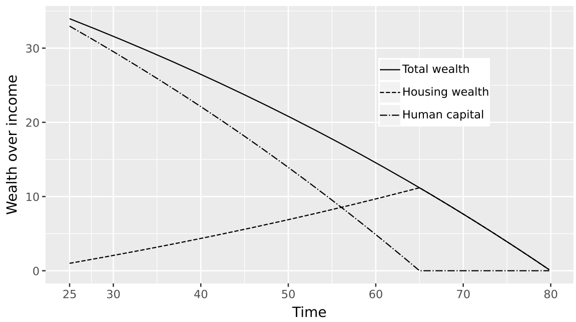
$$v(c^{\star}_t, \tilde{\theta}, N_t) = \frac{W_t + Y_t \mathcal{D}_{r+\sigma_Y\lambda_Y-\mu_Y,\max(T_R-t,0)}}{N_t \mathcal{D}_{r+\sigma_Y\lambda_Y-\mu_Y,T-t}}.$$

As illustrated in Figure 4, investors accumulate housing wealth up to retirement and decumulate afterwards. Housing wealth offsets the early depletion of human capital and ensures a constant bundle consumption rate throughout the lifecycle.

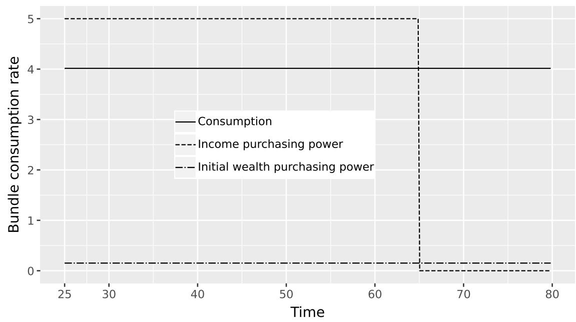
Housing can be understood in this context as a claim to a labor income stream that makes it tradeable in financial markets. Note however that ultimately this is an overly simplified scenario for an investor with some plausible but opinionated beliefs.

In contrast to the continuous trade allowed in this section, later I examine in Section 3 how would policies change if investors can were restricted to trading only at the initial time point. The solution to that static formulation under incomplete markets is very similar.

Figure 4: Lifecycle under extreme risk aversion



(a) Wealth over income



(b) Bundle consumption rate

For an extreme risk averse investor with remaining lifetime of 55 years, 40 working years left and initial wealth equivalent to 1 year of income. Other parameters are described in Table 2.

2.1 Welfare implications

To evaluate the optimal investment strategies, we can measure the welfare gains enabled by the optimal strategy in comparison to other policies, and in particular to the policy that assumes no cointegration. Additionally, we can also assess whether simplifications that assume constant wedges are acceptable as argued in Remark 1.

Welfare changes are measured here based on a strategy equivalent wealth (EW) concept instead of certainty equivalent terms. Using certainty equivalents is customary in the financial literature but it poses some problems in this specific setting. Asking for a certain amount of money as compensation to remove investment risk leaves unaddressed the issue of consumption price risk and the utility derived from consumption. Consumption prices can change from the beginning to the end of the investment horizon. If markets are incomplete and prices cannot be completely hedged, terminal consumption derived from a certain compensation amount is uncertain. The strategy equivalent wealth imposes weaker restrictions and asks for the certain compensation amount needed to replace one investment strategy with another one while providing an equivalent level of expected utility.

Definition 1 (Strategy equivalent wealth)**.** *The strategy equivalent wealth* EW *is the minimum amount of initial wealth* $\tilde{W}$ *needed to compensate the substitution of the lottery associated to investment strategy* π *over initial wealth* W_t *by the forced choice of investment strategy* $\bar{\pi}$*.*

$$\begin{aligned} \mathrm{EW}(\pi, \bar{\pi}, W_t) = \arg\min_{\tilde{W}} \; & \tilde{W} \\ s.t. \; & U(\bar{\pi}, \tilde{W}) \geq U(\pi, W_t). \end{aligned} \tag{50}$$

Lemma 8 (Strategy equivalent wealth)**.** *Suppose that the expected utility* $U(\pi, W_t)$ *is, for any* π*, continuous and strictly monotonically increasing in initial wealth* W_t*, and the range of the function does not depend on* π*. Then the strategy equivalent wealth in the sense of Definition 1 corresponds to*

$$\mathrm{EW}(\pi, \bar{\pi}, W_t) = (U(\bar{\pi}, \cdot))^{-1} \left(U(\pi, W_t)\right). \tag{51}$$

Proof. See Section A.16 □

Lemma 9 (Expected utility with incomplete dynamic markets)**.** *Suppose that an individual applies investment strategy π, which is not necessarily optimal and may depend on state process X_t. Expected utility in dynamic setting* (35) *is*

$$U(t, W_t, X_t, P_t; \pi) = \frac{W_t^{1-\gamma}}{1-\gamma} f(t, X_t, P_t; \pi)^{\gamma}$$

where

$$f(t, X_t, P_t; \pi) = \left(e^{(1-\theta^\intercal \mathbb{1})\log(1-\theta^\intercal \mathbb{1}) + \theta^\intercal \log(\theta) - \theta^\intercal \log(P_t)}\right)^{\frac{1}{\gamma}-1} h(t, X_t; T, \theta, \pi)$$

and $h(t, X_t)$ solves the following PDE

$$\begin{aligned}
\frac{\partial h}{\partial t} =& h(t, X_t; T, \theta, \pi) \left(\frac{\delta}{\gamma} - \left(\frac{1}{\gamma} - 1\right) \begin{pmatrix} r + (\mu_A - r\mathbb{1})^\intercal \pi - \frac{1}{2}\gamma \pi^\intercal \Sigma_A \Sigma_A^\intercal \pi \\ - \theta^\intercal \left(\mu_P + (1-\gamma)\Sigma_P \rho_{PA} \Sigma_A^\intercal \pi\right) \\ + \frac{1}{2}(1-\gamma)\theta^\intercal \Sigma_P \Sigma_P^\intercal \theta + \frac{1}{2} \operatorname{tr}\left(\operatorname{diag}(\theta) \Sigma_P \Sigma_P^\intercal\right) \end{pmatrix} \right) \\
& - h_X^\intercal \left(\mu_X + (1-\gamma)\Sigma_X \left(\rho_{XA}\Sigma_A^\intercal \pi - \rho_{XP}\Sigma_P^\intercal \theta\right)\right) \\
& + \frac{1}{2}(1-\gamma)\frac{1}{h(t, X_t; T, \theta, \pi)} h_X^\intercal \Sigma_X \Sigma_X^\intercal h_X - \frac{1}{2}\operatorname{tr}\left(h_{XX^\intercal}\Sigma_X \Sigma_X^\intercal\right)
\end{aligned} \tag{52}$$

with boundary condition $h(T, X_t; T, \theta, \pi) = 1$.

Proof. See Section A.17 □

The PDE in (52) is an instance of the generic PDE described in (3) and inherits all of its properties and solutions. Some remarks about this relationship can be found at the end of Section A.17. Substituting $\pi_t^\star$ from (36) into the PDE for expected utility (52) coincides with the PDE for indirect utility (38).

Corollary 1 (Equivalent wealth with incomplete dynamic markets)**.** *The strategy equivalent wealth* EW *in dynamic setting* (35) *is*

$$\mathrm{EW}(\pi, \bar{\pi}, W_t) = W_t \left(\frac{h(t, X_t; T, \theta, \pi)}{h(t, X_t; T, \theta, \bar{\pi})}\right)^{\frac{\gamma}{1-\gamma}}.$$

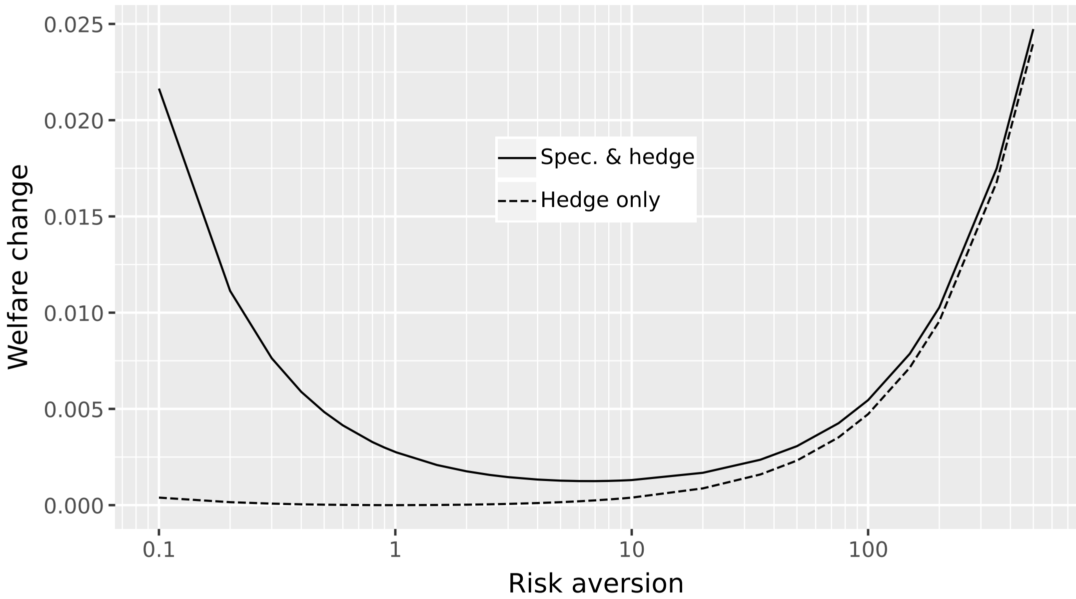
Proof. Straightforward application of Lemma 8 to expected utility from Lemma 9. □

In the remainder of this analysis, welfare effects are measured in annualized growth rates as this magnitude has the same interpretation for any time horizon

$$\text{Annualized welfare growth} = \frac{1}{T}\log\left(\frac{\mathrm{EW}(\pi, \bar{\pi}, W_t)}{W_t}\right).$$

To accommodate investment strategies π, the state process X_t can include additional components not directly related to investment assets or consumption prices. For instance, it may include an exponential process of time, like the one used by strategies (41) and (42), which has linear dynamics with constant parameters.

Figure 5: Annualized welfare gains in dynamic setup



Annualized welfare gains of using the optimal investment strategy (42) in an scenario with cointegration, compared to using the investment strategy of a model that cannot capture non-cointegration (39). “Spec. & hedge” parameters are described in Table 2. “Hedge only” shows the impact attributatable to hedging components by overriding parameters $\lambda_Y = \lambda_N = 0$.

Annualized welfare gains in Figure 5 range from 0.12% for investors with a risk aversion coefficient of 7, to larger gains of about 2.4% for investors who are comparatively more risk tolerant or more risk averse. Notice that an annualized 0.12% can compound to a non-negligible 4.8% over a time horizon of 40 years.Overall welfare changes conflate speculative and hedging motives, as investors take advantage of the orthogonal dependence structure to foster diversification in their speculative positions. When looking at gains attributable to hedging components, it is clear that risk averse investors benefit the most while there is no change for log-investors, who have a risk aversion coefficient of 1.

These figures give us a rough idea about the impact magnitude that the presence of cointegration between rent prices and labor income can have on the optimal strategy and welfare effects. Please be aware that parameters from Table 2 were hand-picked to illustrate the possibilities afforded by the mechanism embedded in this model and were not calibrated from data.

3 Static portfolio optimization

This section analyzes the static investment problem and compares results with those of the dynamic investment problem obtained in Section 2. The purpose of this simpler problem formulation is to double check the basic results of its richer dynamic counterpart.

Consider the problem of an expected utility maximizer with a CRRA utility (2) over a Cobb-Douglas consumption bundle (4) to be bought at time T

$$\sup_{\pi} U(\pi, W_t) \quad \text{where} \quad U(\pi, W_t) = E\left[u(v(W_T, \theta, P_T))\right].$$

Consumption is paid with the proceeds resulting from the accumulated wealth process W_t and controlled through the static investment strategy π fixed at initial time t

$$\frac{\mathrm{d}W_t}{W_t} = (\pi^\intercal(\mu_A - r_t\mathbb{1}) + r_t)\,\mathrm{d}t + \pi^\intercal\Sigma_A\,\mathrm{d}Z_{A,t}\,.$$

The state process X_t described by

$$\mathrm{d}X_t = (\alpha_X - \mathrm{diag}(\beta_X)X_t)\,\mathrm{d}t + \Sigma_X\,\mathrm{d}Z_{X,t}$$

determines the risk free interest rate r_t, the risk premium $\mu_A - r_t\mathbb{1}$ and the drift of consumption prices μ_P

$$\begin{aligned} r_t =& \alpha_r + \beta_r^\intercal X_t \\ \mu_A - r_t\mathbb{1} =& \alpha_\Pi + \beta_\Pi X_t \\ \mu_P =& \alpha_P + \beta_P X_t. \end{aligned}$$

Markets need not be complete and consumption prices P_t satisfy the dynamics below

$$\frac{\mathrm{d}P_t}{P_t} = \mu_P\,\mathrm{d}t + \Sigma_P\,\mathrm{d}Z_{P,t}\,.$$

Diffusion matrices, as well as correlations between the Wiener processes, are assumed to be constant.

Theorem 1 (Static portfolio optimization)**.** *The optimal investment strategy, if the matrix ψ_2 is positive definite, is*

$$\pi^\star = \left(\frac{\psi_2^\intercal + \psi_2}{2}\right)^{-1}\psi_1 \tag{53}$$

where

$$\begin{aligned} \psi_1 =& \alpha_\Pi(T-t) + \beta_\Pi\left(\frac{\alpha_X}{\beta_X}(T-t) + \mathrm{diag}\left(\frac{1-e^{-\beta_X(T-t)}}{\beta_X}\right)\left(X_t - \frac{\alpha_X}{\beta_X}\right)\right) \\ &+ (\gamma-1)\left(\begin{array}{l} \left(\beta_\Pi \,\mathrm{Var}\left[\int_t^T X_s\,\mathrm{d}s\right] + \Sigma_A \,\mathrm{Cov}\left[\int_t^T X_s\,\mathrm{d}s\,, \int_t^T \mathrm{d}Z_{A,s}\right]^\intercal\right)(\beta_P^\intercal\theta - \beta_r) \\ + \left(\Sigma_A\rho_{PA}^\intercal\Sigma_P^\intercal(T-t) + \beta_\Pi \,\mathrm{Cov}\left[\int_t^T X_s\,\mathrm{d}s\,, \int_t^T \mathrm{d}Z_{P,s}\right]\Sigma_P^\intercal\right)\theta \end{array}\right) \\ \psi_2 =& \gamma\Sigma_A\Sigma_A^\intercal(T-t) + (\gamma-1)\beta_\Pi\left(\mathrm{Var}\left[\int_t^T X_s\,\mathrm{d}s\right]\beta_\Pi^\intercal + 2\,\mathrm{Cov}\left[\int_t^T X_s\,\mathrm{d}s\,, \int_t^T \mathrm{d}Z_{A,s}\right]\Sigma_A^\intercal\right) \end{aligned}$$

Sufficient conditions for the inverted matrix ψ_2 to be positive definite are $\gamma \geq 1$ or $\beta_\Pi = 0$. Expected utility $U(\pi, W_t)$ is given in (A.54) *and indirect utility can be obtained by plugging $\pi^\star$ into it. Closed form expressions for expectations and covariances related to the state process X_t are available in* (A.50) *to* (A.53)*.*

Proof. See Section A.18. □

Let analyze the solution to our simple lifecycle model with housing derived from the static problem formulation. The optimal fraction to invest in stocks $\pi_S^\star$ is identical to its dynamic counterpart (43) in all of its variants. In the case of no cointegration, the optimal investment fraction for rents $\pi_N^\star$ also coincides with the dynamic strategy (39). Slight differences begin to appear in the cointegration model with complete housing markets: while $\pi_Y^\star$ coincides with (40), the investment fraction exposed to rent wedge ν_N risk becomes

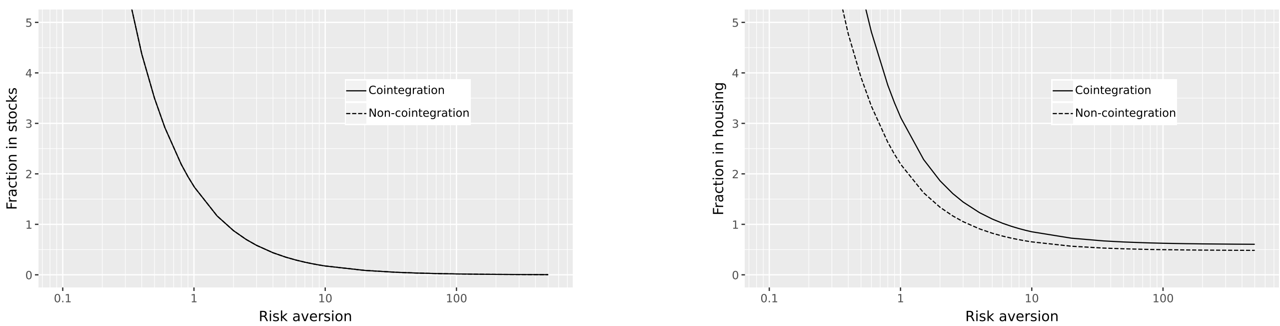
$$\pi_{\nu_N}^\star = e^{\kappa_{\nu_N} h}\left(\frac{1}{\gamma}\frac{\lambda_{\nu_N}}{\sigma_{\nu_N}} + \left(1-\frac{1}{\gamma}\right)\frac{\mathcal{D}_{\kappa_{\nu_N},T-t}}{T-t}\theta_N\right). \tag{54}$$

In the case of incomplete housing markets the fraction to invest in rents changes to

$$\pi_N^\star = \frac{1}{\gamma}\frac{\lambda_Y\sigma_Y + \lambda_{\nu_N}\sigma_{\nu_N}e^{-\kappa_{\nu_N}h}}{\sigma_Y^2+\sigma_{\nu_N}^2 e^{-2\kappa_{\nu_N}h}} + \left(1-\frac{1}{\gamma}\right)\left(\frac{\sigma_Y^2+\sigma_{\nu_N}^2 e^{-\kappa_{\nu_N}h}\frac{\mathcal{D}_{\kappa_{\nu_N},T-t}}{T-t}}{\sigma_Y^2+\sigma_{\nu_N}^2 e^{-2\kappa_{\nu_N}h}}\theta_N + \frac{\sigma_Y^2}{\sigma_Y^2+\sigma_{\nu_N}^2 e^{-2\kappa_{\nu_N}h}}\theta_L\right). \tag{55}$$

Differences are driven mostly by hedging demand. Whereas the dynamic strategy considered instantaneous exposure $e^{-\kappa_{\nu_N}(T-t)}$ to rent wedge ν_N risk, the static strategy considers average exposure $\frac{\mathcal{D}_{\kappa_{\nu_N},T-t}}{T-t}$ over the investment horizon. That said, these differences are negligible when $\nu_N h$ is large, which strengthens the cointegration effect.

Figure 6: Investment strategies



(a) Fraction in stocks

(b) Fraction in housing

Parameters are described in Table 2.

Assessing welfare changes using Corollary 2 produces equivalent wealth growth rates that are linear in the time horizon through ψ_2 and quadratic in the strategies π and $\bar{\pi}$.

Corollary 2 (Strategy equivalent wealth)**.** *The strategy equivalent wealth applicable to the static investment problem can be written using* $\pi^\star$ *from* (53) *as*

$$\mathrm{EW}(\pi,\bar{\pi},W_t) = W_t e^{(\pi-\bar{\pi})^\intercal \frac{\psi_2+\psi_2^\intercal}{4}(2\pi^\star-\pi-\bar{\pi})}. \tag{56}$$

Proof. Straightforward application of Lemma 8 to expected utility from (A.54) yields the expression below and it is trivial to verify that (56) is equivalent to it

$$\mathrm{EW}(\pi,\bar{\pi},W_t) = W_t e^{\pi^\intercal\psi_1 - \frac{\pi^\intercal\psi_2\pi}{2} - \bar{\pi}^\intercal\psi_1 + \frac{\bar{\pi}^\intercal\psi_2\bar{\pi}}{2}}.$$

□

Resulting welfare changes are virtually identical to those described in Figure 5 of Section 2.1.

4 Conclusion

This lifecycle model can partly rationalize why some investors who are not attracted to the stock market, may be attracted residential real estate investments even if the risk premium is comparably worse when adjusted for risk. When house prices are cointegrated with income and under certain assumptions, buying a house buys more than just future rents as argued by Sinai and Souleles (2005), it is roughly comparable to buying future labor income. The portfolio problem captures many interesting mechanisms analytically. For instance, it can show why for young investors with a lot of human captial investing into housing leads to income risk overexposure as argued by Kueng et al. (2024), and that it is more advantageous to build up house equity progresively as human capital gets depleated. It also opens the possibility for investors to use residential real estate investments to hedge labor intensive services like healthcare or elderly assistance, which become increasingly important as one ages.

These results could have implications also for pension systems. Pay-as-you-go schemes use labor contributions of current workers to pay retirement benefits to current retirees, while defined contribution schemes and individual pension plans use workers contributions and their accumulated investment returns to pay for their own pension. The direct link between current contributions and benefits in pay-as-you-go provides an automatic hedge against the cost of labor intensive services like elderly care but this feature is not exclusive to this scheme and can be replicated by investors with access to real estate markets. According to this model, defined contribution schemes and individual pension plans can also provide a similar hedge by investing into residential real estate.

A Appendix

A.1 Proof of optimal consumption bundle, Lemma 1

The degenerate case in which the consumption budget is zero $c = 0$ admits only one solution allocating zero to every product $\xi_i = c - \xi^\intercal P = 0$ and making the objective function zero. The remainder of this proof considers the case of a positive consumption budget $c > 0$.

Another degenerate case is when $\theta_i = 0\,\forall i$ or $\exists \theta_i = 1$ make the problem linear in one product or cash-indexed consumption. All other alternatives drop out and the optimal policy allocates the entire budget to the remaining product $\xi_i = \frac{c}{P_i}$ or to cash-indexed consumption $c - \xi^\intercal P = c$ respectively. The remainder of this proof considers the case of strictly concave problems $\|\theta\| \in (0, 1)$.

The objective function becomes zero whenever a product (including the cash indexed product) with non-zero elasticity is allocated zero consumption. This is clearly suboptimal since we can redistribute a fraction of the budget allocated to other products towards products with zero allocations but non-zero elasticities and make the objective function strictly positive. Consequently, we can restrict our attention to non-zero allocations for products that have non-zero elasticities.

These observations make it possible to reformulate the consumption allocation problem (3) in exponential-logarithm form

$$\max_{\xi\in\mathbb{R}_+^{|P|}} \exp\left((1-\theta^\intercal\mathbb{1})\log(c-\xi^\intercal P)+\sum_{i=1}^{|\xi|}\theta_i\log(\xi_i)\right).$$

Second order conditions are satisfied since the objective function is a monotonic transformation over a strictly concave function. The Hessian matrix of the log-objective is negative definite since it is symmetric and second order (cross-)derivatives are negative. The first order conditions for consumption allocation problem are

$$\xi_i = \frac{\theta_i}{P_i}\frac{(c-\xi^\intercal P)}{(1-\theta^\intercal\mathbb{1})}.$$

Using the equations from the first order conditions we can find this recursive expression for $\xi^\intercal P$

$$\xi^\intercal P = \theta^\intercal\mathbb{1}\frac{c-\xi^\intercal P}{1-\theta^\intercal\mathbb{1}}$$

and rearrange terms to arrive at this other explicit expression

$$\xi^\intercal P = \theta^\intercal\mathbb{1}c.$$

Thus, the optimal allocations to dynamically priced products are given by

$$\xi_i = \frac{\theta_i}{P_i}c,$$

the remainder allocation to cash consumption is

$$c-\xi^\intercal P = (1-\theta^\intercal\mathbb{1})\,c,$$

and the objective function becomes

$$e^{(1-\theta^\intercal\mathbb{1})\log(1-\theta^\intercal\mathbb{1})+\theta^\intercal\log(\theta)-\theta^\intercal\log(P)}c.$$

A.2 Proof of price for terminal payoff, Lemma 2

The price $\Omega(t,X_t,Q_t;T)$ of the claim to a payoff Q_T at terminal date T given the pricing kernel K_t with market price of risk Λ corresponds to

$$\Omega(t,X_t,Q_t;T) = E\left[\frac{K_T}{K_t}Q_T|\mathcal{F}_t\right]$$

and by the law of iterated expectations we have the following recurrence relation for $s\in[t,T]$

$$\Omega(t,X_t,Q_t;T) = E\left[\frac{K_s}{K_t}E\left[\frac{K_T}{K_s}Q_T|\mathcal{F}_s\right]|\mathcal{F}_t\right] = E\left[\frac{K_s}{K_t}\Omega(s,X_s,Q_s;T)|\mathcal{F}_t\right].$$

For a small Δt, we have that

$$\Omega(t,X_t,Q_t;T) = E\left[\frac{K_{t+\Delta t}}{K_t}\Omega(t+\Delta t,X_{t+\Delta t},Q_{t+\Delta t};T)|\mathcal{F}_t\right].$$

Subtracting $\Omega(t, X_t, Q_t; T)$ from both sides, dividing by Δt and taking $\lim_{\Delta t \downarrow 0}$ we obtain

$$
\begin{aligned}
0 =& E\left[\lim_{\Delta t \downarrow 0} \frac{e^{-\int_0^{\Delta t} r_{t+s} + \frac{\Lambda_{t+s}^\intercal \Lambda_{t+s}}{2} \mathrm{d}s - \int_0^{\Delta t} \Lambda_{t+s}^\intercal \mathrm{d}Z_{A,t+s}} - 1}{\Delta t} \Omega(t+\Delta t, X_{t+\Delta t}, Q_{t+\Delta t}; T) | \mathcal{F}_t\right] \\
&+ E\left[\lim_{\Delta t \downarrow 0} \frac{\Omega(t+\Delta t, X_{t+\Delta t}, Q_{t+\Delta t}; T) - \Omega(t, X_t, Q_t; T)}{\Delta t} | \mathcal{F}_t\right]
\end{aligned}
$$

where

$$
\begin{aligned}
&E\left[\lim_{\Delta t \downarrow 0} \frac{e^{-\int_0^{\Delta t} r_{t+s} + \frac{\Lambda_{t+s}^\intercal \Lambda_{t+s}}{2} \mathrm{d}s - \int_0^{\Delta t} \Lambda_{t+s}^\intercal \mathrm{d}Z_{A,t+s}} - 1}{\Delta t} \Omega(t+\Delta t, X_{t+\Delta t}, Q_{t+\Delta t}; T) | \mathcal{F}_t\right] \\
&= -r_t \Omega(t, X_t, Q_t; T) - \Omega_X^\intercal \Sigma_X \rho_{XA} \Lambda - \Omega_Q Q_t \Sigma_Q \rho_{QA} \Lambda
\end{aligned}
$$

and

$$
\begin{aligned}
&E\left[\lim_{\Delta t \downarrow 0} \frac{\Omega(t+\Delta t, X_{t+\Delta t}, Q_{t+\Delta t}; T) - \Omega(t, X_t, Q_t; T)}{\Delta t} | \mathcal{F}_t\right] \\
&= \frac{\partial \Omega}{\partial t} + \Omega_Q^\intercal Q \mu_Q + \Omega_X^\intercal \mu_X + \frac{1}{2} \Omega_{QQ^\intercal} Q_t \Sigma_Q \Sigma_Q^\intercal Q_t + \frac{1}{2} \operatorname{tr}(\Omega_{XX^\intercal} \Sigma_X \Sigma_X^\intercal) + \Omega_{XQ}^\intercal \Sigma_X \rho_{XQ} \Sigma_Q^\intercal Q_t.
\end{aligned}
$$

Substituting the expectations and rearranging terms gives the following PDE

$$
\begin{aligned}
0 =& -\Omega(t, X_t, Q_t; T) r + \Omega_X^\intercal \left(\mu_X - \Sigma_X \rho_{XA} \Lambda\right) + \left(\mu_Q - \Sigma_Q \rho_{QA} \Lambda\right) Q_t \Omega_Q \\
&+ \frac{\partial \Omega}{\partial t} + \frac{1}{2} \operatorname{tr}\left(\Omega_{XX^\intercal} \Sigma_X \Sigma_X^\intercal\right) + \frac{1}{2} \Omega_{QQ} Q_t \Sigma_Q \Sigma_Q^\intercal Q_t + \Omega_{XQ}^\intercal \Sigma_X \rho_{XQ} \Sigma_Q^\intercal Q_t
\end{aligned} \tag{A.1}
$$

with boundary condition $\Omega(T, X_T, Q_T; T) = Q_T$.

Finally, it's straightforward to reformulate this PDE in terms of $\tilde{\Omega}(t, X_t; T) = \Omega(t, X_t, Q_t; T) Q_t^{-1}$ using homogeneity when market parameters do not depend on Q_t

$$
\begin{aligned}
0 =& \frac{\partial \tilde{\Omega}}{\partial t} + \left(\mu_Q - r - \Sigma_Q \rho_{QA} \Lambda\right) \tilde{\Omega}(t, X_t; T) \\
&+ \tilde{\Omega}_X^\intercal \left(\mu_X + \Sigma_X \left(\rho_{XQ} \Sigma_Q^\intercal - \rho_{XA} \Lambda\right)\right) + \frac{1}{2} \operatorname{tr}\left(\tilde{\Omega}_{XX^\intercal} \Sigma_X \Sigma_X^\intercal\right)
\end{aligned}
$$

with boundary condition $\tilde{\Omega}(T, X_T; T) = 1$.

The PDE for $\tilde{\Omega}(t, X_t; T)$ is a particular case of the generic PDE from Lemma 3 parametrized as

$$
g\begin{pmatrix}
t, X_t; \\
R = \mu_Q - r - \Sigma_Q \rho_{QA} \Lambda, \\
B = \mu_X + \Sigma_X \left(\rho_{XQ} \Sigma_Q^\intercal - \rho_{XA} \Lambda\right), \\
C = \Sigma_X \Sigma_X^\intercal, \\
D = \Sigma_X \Sigma_X^\intercal
\end{pmatrix}
$$

with boundary condition $g(T, X_T) = 1$.

Section A.5 explains how to reduce the generic PDE to a system of Riccati ODEs by parametrizing A, B, C, D quadratically, which in this case can be constructed from the following building blocks

$$
\begin{aligned}
\mu_Q &= {}_Q\alpha + {}_Q\beta_p \, X^p + X_p \, {\eta_h}^p \, {}_Q{\omega^h}_m \, {\eta^m}_q \, X^q \\
r &= {}_r\alpha + {}_r\beta_p \, X^p + X_p \, {\eta_h}^p \, {}_r{\omega^h}_m \, {\eta^m}_q \, X^q \\
\Sigma_Q \rho_{QA} \Lambda &= {}_{\Sigma_Q A}\alpha + {}_{\Sigma_Q A}\beta_p \, X^p + X_p \, {\eta_h}^p \, {}_{\Sigma_Q A}{\omega^h}_m \, {\eta^m}_q \, X^q \\
(\mu_X)^k &= {}_X\alpha^k - {}_X{\beta^k}_p \, X^p + X_p \, {\eta_h}^p \, {}_X{\omega^{kh}}_m \, {\eta^m}_q \, X^q \\
\left(\Sigma_X \rho_{XQ} \Sigma_Q^\intercal\right)^k &= {}_{\Sigma_X \Sigma_Q}\alpha^k + {}_{\Sigma_X \Sigma_Q}{\beta^k}_p \, X^p + X_p \, {\eta_h}^p \, {}_{\Sigma_X \Sigma_Q}{\omega^{kh}}_m \, {\eta^m}_q \, X^q \\
(\Sigma_X \rho_{XA} \Lambda)^k &= {}_{\Sigma_X A}\alpha^k + {}_{\Sigma_X A}{\beta^k}_p \, X^p + X_p \, {\eta_h}^p \, {}_{\Sigma_X A}{\omega^{kh}}_m \, {\eta^m}_q \, X^q \\
{(\Sigma_X \Sigma_X^\intercal)^k}_l &= {}_{\Sigma_X}{\alpha^k}_l + {}_{\Sigma_X}{\beta^k}_{lp} \, X^p + X_p \, {\eta_h}^p \, {}_{\Sigma_X}{\omega^k{}_l{}^h}_m \, {\eta^m}_q \, X^q
\end{aligned}
$$

Section A.6 shows how to explicitly solve the diagonalized version of the aforementioned Riccati ODEs for an ample range of cases.

A.3 Proof of replicating strategy for terminal payoff, Lemma 2

Suppose that markets are complete and we want to find the minimum initial capital $\Omega(t, X_t, Q_t; T)$ necessary to make sure that at a terminal date T the value of our portfolio is equals the desired payoff $\Omega(T, X_T, Q_T; T) = Q_T$. Changes in the value of the replicating portfolio reflect the proceeds from investing funds according to the investment strategy π_t

$$
\mathrm{d}\Omega_t = \Omega(t, X_t, Q_t; T) \left(\pi_t^\intercal (\mu_A - r\mathbb{1}) + r\right) \mathrm{d}t + \Omega(t, X_t, Q_t; T) \pi_t^\intercal \Sigma_A \, \mathrm{d}Z_{A,t} \tag{A.2}
$$

Assuming that X_t and Q_t are tradeable through A_t and applying Itô's lemma to $\Omega(t, X_t, Q_t; T)$ shows that

$$
\begin{aligned}
\mathrm{d}\Omega_t = &\frac{\partial \Omega}{\partial t} \, \mathrm{d}t + \Omega_X^\intercal \left(\mu_X \, \mathrm{d}t + \Sigma_X \rho_{XA} \, \mathrm{d}Z_{A,t}\right) + \Omega_Q Q_t \left(\mu_Q \, \mathrm{d}t + \Sigma_Q \rho_{QA} \, \mathrm{d}Z_{A,t}\right) \\
&+ \frac{1}{2} \operatorname{tr}\left(\Omega_{XX^\intercal} \Sigma_X \Sigma_X^\intercal\right) \mathrm{d}t + \frac{1}{2} Q_t^2 \Omega_{QQ} \Sigma_Q \Sigma_Q^\intercal \, \mathrm{d}t + Q_t \Omega_{XQ}^\intercal \Sigma_X \rho_{XQ} \Sigma_Q^\intercal \, \mathrm{d}t
\end{aligned} \tag{A.3}
$$

where I have used $\mathrm{d}Z_{Q,t} = \rho_{QA} \, \mathrm{d}Z_{A,t}$ and $\mathrm{d}Z_{X,t} = \rho_{XA} \, \mathrm{d}Z_{A,t}$. There is a unique strategy π_t that deterministically replicates the desired dynamics by cancelling out the Brownian motions on both sides of (A.3) when $\mathrm{d}\Omega_t$ is replaced by (A.2)

$$
\pi_t = (\Sigma_A^\intercal)^{-1} \, \frac{\rho_{XA}^\intercal \Sigma_X^\intercal \Omega_X + \rho_{QA}^\intercal \Sigma_Q^\intercal Q_t \Omega_Q}{\Omega(t, X_t, Q_t; T)}
$$

Using this investment strategy, the required initial capital $\Omega(t, X_t, Q_t; T)$ is the solution to the following PDE

$$
\begin{aligned}
0 = &- \Omega(t, X_t, Q_t; T) r + \Omega_X^\intercal \left(\mu_X - \Sigma_X \rho_{XA} \Sigma_A^{-1} (\mu_A - r\mathbb{1})\right) \\
&+ \left(\mu_Q - \Sigma_Q \rho_{QA} \Sigma_A^{-1} (\mu_A - r\mathbb{1})\right) Q_t \Omega_Q \\
&+ \frac{\partial \Omega}{\partial t} + \frac{1}{2} \operatorname{tr}\left(\Omega_{XX^\intercal} \Sigma_X \Sigma_X^\intercal\right) + \frac{1}{2} \Omega_{QQ} Q_t \Sigma_Q \Sigma_Q^\intercal Q_t + \Omega_{XQ}^\intercal \Sigma_X \rho_{XQ} \Sigma_Q^\intercal Q_t
\end{aligned} \tag{A.4}
$$

with boundary condition $\Omega(T, X_T, Q_T; T) = Q_T$.

This equation coincides with (A.1). Note that $\Lambda = \Sigma_A^{-1}(\mu_A - r\mathbb{1})$ is the unique market price of risk satisfying (1) when markets are complete.

A.4 Brief explanation of tensor notation

A tensor is a generalization of vectors and matrices that can accommodate any arbitrary number of axes.

Consider matrices A and B of size $m \times n$, column vectors x, y and z of size n and a scalar c. Using tensor notation we need refer to the matrix $A^i{}_j$ specifying the indices i for the row axis and j for the column axis. The position of indices determines the axes that they refer to, for instance $A^j{}_i$ refers to rows with j and to columns with i but it is otherwise equivalent to our previous example except for the change in "labels". If an index is omitted, it is understood that we contract the omitted axis and sum its values. E.g. writing A^i in tensor notation would be equivalent to $A\mathbb{1}$ in matrix notation.

Notice that indices need to be specified in both sides of an assignment and that they can appear as subscripts (covariant) or superscripts (contravariant). When the same index appears twice in a product expression once as a subscript and once as a superscript regardless of product order, it is equivalent to the inner product, that is

$$y^i \;=\; A^i{}_j\, x^j = x^j\, A^i{}_j \;=\; A^{ij}\, x_j = x_j\, A^{ij}$$

is equivalent to $y = Ax$ in matrix notation. If an index appears twice as a superscript or subscript on different tensors regardless of product order, then it is equivalent to an elementwise product, e.g. $z^i = x^i y^i = y^i x^i$ is equivalent to $z = \operatorname{diag}(x)\, y$. The outer product $A = xy^\intercal$ in matrix notation corresponds to $A^i{}_j \;=\; x^i\, y_j \;=\; y_j\, x^i$ in tensor notation. If an unassigned index appears twice on the same tensor, once as covariant and once as contravariant $c = A^i{}_i$, it is equivalent to the trace in matrix notation $c = \operatorname{tr}(A)$. An assigned index appearing twice as covariant or contravariant on the same tensor $x^i \;=\; A^{ii}$ extracts its diagonal $x = \operatorname{diag}(A)$. With these rules it is also easy to see that the matrix transpose $B = A^\intercal$ is equivalent to $B^i{}_j \;=\; A_j{}^i$.

In this environment it is handy to define the Kronecker delta δ

$$\delta^i{}_j \;=\begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}, \tag{A.5}$$

then building a diagonal matrix from a vector $A = \operatorname{diag}(x)$ becomes $A^i{}_j \;=\; x^i\, \delta^i{}_j$.

Tensor notation is very flexible and allows some expressions that cannot be translated to matrix notation. While in matrices the first axis is contravariant and the second axis is covariant, tensor notation allows the expression A_{ij} making both indices covariant. Finally, the rules explained here extend quite naturally to tensors of more than 2 axes, like $C^i{}_{jk}{}^l$.

A.5 Reducing generic PDE to Riccati ODEs, Lemma 3

Consider the PDE (14) with some deterministic boundary condition $g(T, X_t)$ and dynamic parameters R as a scalar, B of size $|X|$, C symmetric of size $|X| \times |X|$ and D symmetric of size $|X| \times |X|$

$$0 = \frac{\partial g}{\partial t} + g(t, X_t)R + g_X^\intercal B + \frac{1}{2}\frac{g_X^\intercal\,(C - D)\, g_X}{g(t, X_t)} + \frac{1}{2}\operatorname{tr}\left(g_{XX^\intercal} D\right).$$

To reduce the PDE (14) into Riccati ODEs, it can be reparametrized using quadratic terms as in Liu (2007) and expressed in tensor notation[2]

$$
\begin{aligned}
R &= {}_R\alpha + {}_R\beta_p\, X^p + X_p\, \eta_h{}^p\, {}_R\omega^h{}_m\, \eta^m{}_q\, X^q \\
B^k &= {}_B\alpha^k + {}_B\beta^k{}_p\, X^p + X_p\, \eta_h{}^p\, {}_B\omega^{kh}{}_m\, \eta^m{}_q\, X^q \\
C^k{}_l &= {}_C\alpha^k{}_l + {}_C\beta^k{}_{lp}\, X^p + X_p\, \eta_h{}^p\, {}_C\omega^k{}_l{}^h{}_m\, \eta^m{}_q\, X^q \\
D^k{}_l &= {}_D\alpha^k{}_l + {}_D\beta^k{}_{lp}\, X^p + X_p\, \eta_h{}^p\, {}_D\omega^k{}_l{}^h{}_m\, \eta^m{}_q\, X^q
\end{aligned}
$$

where parameters subscripted with $_C$ and $_D$ are symmetric with respect to the first two indices, e.g. ${}_C\beta^k{}_{lp} = {}_C\beta_l{}^k{}_p$.

This general formulation includes those of Kim and Omberg (1996), Wachter (2002) and Liu (2007). The solution to the PDE is based upon the following ansatz

$$g(t,X) = e^{a(T-t) + X_i\, b(T-t)^i + \frac{1}{2} X_i\, \eta_\mu{}^i\, c(T-t)^\mu{}_\nu\, \eta^\nu{}_j\, X^j}$$

assuming that the boundary condition can be decomposed as

$$g(T,X) = e^{a(0) + X_i\, b(0)^i + \frac{1}{2} X_i\, \eta_\mu{}^i\, c(0)^\mu{}_\nu\, \eta^\nu{}_j\, X^j} .$$

Main Riccati ODEs Using $\tau = T - t$ and substituting the ansatz into the PDE (14) yields, under restrictions (A.9)-(A.11) and (A.12) detailed below, the following system of coupled ODEs[3]

$$
\begin{aligned}
\frac{\partial a}{\partial \tau} &= {}_R\alpha + {}_B\alpha^k\, b(\tau)_k + \frac{1}{2}\, {}_C\alpha^k{}_l\, b(\tau)_k\, b(\tau)^l + \frac{1}{2}\, {}_D\alpha^k{}_l\, \eta_\mu{}^l\, \eta^\nu{}_k\, \frac{c(\tau)^\mu{}_\nu + c(\tau)_\nu{}^\mu}{2} \qquad \text{(A.6)} \\
\frac{\partial b^i}{\partial \tau} &= {}_R\beta^i + {}_B\beta^{ki}\, b(\tau)_k + \frac{1}{2}\, {}_C\beta^k{}_l{}^i\, b(\tau)_k\, b(\tau)^l + {}_C\alpha^k{}_l\, \eta^\nu{}_k\, \eta_\mu{}^i\, \frac{c(\tau)^\mu{}_\nu + c(\tau)_\nu{}^\mu}{2}\, b(\tau)^l \\
&\quad + \left({}_B\alpha^k\, \eta_\mu{}^i + \frac{1}{2}\, {}_D\beta^k{}_l{}^i\, \eta_\mu{}^l \right) \eta^\nu{}_k\, \frac{c(\tau)^\mu{}_\nu + c(\tau)_\nu{}^\mu}{2} \qquad \text{(A.7)} \\
\frac{\partial c^i{}_j}{\partial \tau} &= 2\, {}_R\omega^i{}_j + 2\, {}_B\hat\beta_j{}^\nu\, \frac{c(\tau)^i{}_\nu + c(\tau)_\nu{}^i}{2} + {}_D\omega^k{}_l{}^i{}_j\, \eta_\mu{}^l\, \eta^\nu{}_k\, \frac{c(\tau)^\mu{}_\nu + c(\tau)_\nu{}^\mu}{2} \\
&\quad + {}_C\alpha^k{}_l\, \eta^\nu{}_k\, \eta_\mu{}^l\, \frac{c(\tau)^i{}_\nu + c(\tau)_\nu{}^i}{2}\, \frac{c(\tau)^\mu{}_j + c(\tau)_j{}^\mu}{2} + 2\, {}_B\omega^{ki}{}_j\, b(\tau)_k + {}_C\omega^k{}_l{}^i{}_j\, b(\tau)^l\, b(\tau)_k \\
&\qquad \text{(A.8)}
\end{aligned}
$$

with $a(0)$, $b(0)^i$, $c(0)^i{}_j$ as boundary conditions.

The following restrictions removed the cubic and quartic X terms from the PDE

$$ {}_B\omega^{kh}{}_m\, \eta^\nu{}_k\, \frac{c(\tau)^\mu{}_\nu + c(\tau)_\nu{}^\mu}{2} = 0 \qquad \text{(A.9)}$$

$$ {}_C\omega^k{}_l{}^h{}_m\, b(\tau)^l\, \eta^\nu{}_k\, \frac{c(\tau)^\mu{}_\nu + c(\tau)_\nu{}^\mu}{2} = 0 \qquad \text{(A.10)}$$

[2]See Section A.4 for a brief introduction to tensor notation

[3]The ODE for $c^i{}_j$ admits an alternative definition to (A.8), where the term ${}_B\hat\beta_j{}^\nu\, \frac{c(\tau)^i{}_\nu + c(\tau)_\nu{}^i}{2}$ is replaced by ${}_B\hat\beta^i{}_\mu\, \frac{c(\tau)^\mu{}_j + c(\tau)_j{}^\mu}{2}$. The definition of choice is simply a matter of convetion and both produce the same final expression $g(t, X_t)$ because the quadratic term is symmetric. Some terms related to C were also simplified relying on its symmetry.

$$
{}_C\beta^{k\,p}_{\ \ l}\ \eta^{\nu}_{\ k}\ \frac{c(\tau)^{\mu}_{\ \nu} + c(\tau)_{\nu}^{\ \mu}}{2} = 0 \tag{A.11}
$$

while additionally imposing this other restriction helped to simplify the ODE for $c^{i}_{\ j}$ in (A.8)

$$
{}_B\beta^{k}_{\ q}\ \eta^{\nu}_{\ k}\ \frac{c(\tau)^{i}_{\ \nu} + c(\tau)_{\nu}^{\ i}}{2} = {}_B\hat{\beta}_j^{\ \nu}\ \eta^{j}_{\ q}\ \frac{c(\tau)^{i}_{\ \nu} + c(\tau)_{\nu}^{\ i}}{2} \tag{A.12}
$$

where ${}_B\hat{\beta}_j^{\ \nu}$ is defined implicitly through this restriction.

A solution to a simple case In the special case of ${}_R\beta^i = 0$, ${}_R\omega^i_{\ j} = 0$, $b(0)^i = 0$, $c(0)^i_{\ j} = 0$, the restrictions (A.9)-(A.11) and (A.12) are not needed, since those terms involve $c(\tau)^i_{\ j}$ and its value is zero. The solution in this case is simply

$$
\begin{aligned}
a(\tau) &= a(0) + {}_R\alpha\,\tau \\
b(\tau)^i &= 0 \\
c(\tau)^i_{\ j} &= 0.
\end{aligned}
$$

Diagonalized Riccati ODEs Some interesting closed form solutions can be obtained by diagonalization. Let δ denote the Kronecker delta (A.5) and the binary operator $\odot$ denote the Hadamard product. The Riccati ODEs can be diagonalized with $c(\tau) = \operatorname{diag}(\tilde{c}_1(\tau), \tilde{c}_2(\tau), \ldots)$ to partly decouple the ODEs as

$$
\begin{aligned}
\frac{\partial a}{\partial \tau} &= \upsilon_1 + \upsilon_2^{\intercal} b(\tau) + \frac{1}{2}\operatorname{tr}\left(\ell_1 b(\tau) b(\tau)^{\intercal}\right) + \frac{1}{2}\operatorname{tr}\left(\ell_2 \operatorname{diag}\left(\tilde{c}(\tau)\right)\right) \\
\frac{\partial b_i}{\partial \tau} &= \upsilon_{3,i} - \upsilon_{4,i} b(\tau)_i + \frac{1}{2}\upsilon_{5,i} b_i^2(\tau) + \upsilon_{8,i}\tilde{c}_i(\tau) + \upsilon_{6,i}\tilde{c}_i(\tau) b_i(\tau) \\
\frac{\partial \tilde{c}_i}{\partial \tau} &= 2\upsilon_{7,i} - 2\upsilon_{9,i}\tilde{c}_i(\tau) + \upsilon_{6,i}\tilde{c}_i^2(\tau)
\end{aligned}
$$

with $a(0)$, $b_i(0)$ and $\operatorname{diag}(\tilde{c}(0)) = c(0)$ as boundary conditions where scalar parameter υ_1, vector parameters $\upsilon_2, \ldots, \upsilon_9$ and matrix parameters ℓ_1, ℓ_2 correspond, in matrix notation, to the following decomposition

$$
\begin{aligned}
R &= \upsilon_1 + \upsilon_3^{\intercal} X + \upsilon_7^{\intercal} X^2 \\
B &= \upsilon_2 - \upsilon_4 \odot X \\
C &= \ell_1 + \operatorname{diag}(\upsilon_5 \odot X) \\
D &= \ell_2 + \operatorname{diag}(\varpi_1 \odot X) + \operatorname{diag}\left(\varpi_2 \odot X^2\right) \\
\upsilon_6 &= \operatorname{diag}(\ell_1) \\
\tilde{\upsilon}_7 &= \mathbb{1}_{\upsilon_7 \neq 0\ \vee\ \tilde{c}(0) \neq 0}
\end{aligned}
$$

subject to restrictions

$$
\upsilon_5 \odot \tilde{\upsilon}_7 = 0 \qquad \tilde{\upsilon}_7 \mathbb{1}^{\intercal} \odot \ell_1 = \tilde{\upsilon}_7 \mathbb{1}^{\intercal} \odot \ell_1^{\intercal} = \operatorname{diag}(\tilde{\upsilon}_7 \odot \upsilon_6).
$$

The diagonalized ODEs above admit a more general decomposition of D, although it requires the use of tensor notation

$$
\begin{aligned}
D^{k}_{\ l} &= (\ell_2)^{k}_{\ l} + {}_D\beta^{k\,i}_{\ \ l}\ X_i + {}_D\omega^{k\,i}_{\ \ l\,j}\ X_i\ X^j \\
(\varpi_1)_i &= {}_D\beta_{iii} \\
(\varpi_2)_i &= {}_D\omega_{iiii}
\end{aligned}
$$

subject to restrictions

$$
\begin{aligned}
(\tilde{\upsilon}_7)^k\ (\tilde{\upsilon}_7)^i\ {}_D\beta^{kki} &= \delta^{ki}\ (\tilde{\upsilon}_7)^i\ (\varpi_1)^i \\
(\tilde{\upsilon}_7)^k\ (\tilde{\upsilon}_7)^i\ (\tilde{\upsilon}_7)_j\ {}_D\omega^{kki}{}_j &= \delta^{ki}\ \delta^i{}_j\ (\tilde{\upsilon}_7)^i\ (\varpi_2)^i\,.
\end{aligned}
$$

Remaining parameters are

$$
\upsilon_{8,i} = \upsilon_{2,i} + \frac{\varpi_{1,i}}{2} \qquad \upsilon_{9,i} = \upsilon_{4,i} - \frac{\varpi_{2,i}}{2}.
$$

Section A.6 describes how to solve these diagonalized Riccati ODEs explicitly, which in turn solve the PDE as

$$
g(t,X) = e^{a(T-t)+b(T-t)^\intercal X + \frac{1}{2}\tilde{c}(T-t)^\intercal X^2}.
$$

The diagonalized Riccati ODEs were derived by imposing some restrictions to the main Riccati ODEs above. Consider the case in which $\eta = I$ is the identity matrix, ${}_B\omega^{ki}{}_j = 0$, ${}_C\omega^{k}{}_l{}^{i}{}_j = 0$, the following diagonal restrictions hold

$$
\begin{aligned}
{}_B\beta^k{}_p &= {}_B\hat{\beta}^k{}_p = -\,\delta^k{}_p\ (\upsilon_4)^k \\
{}_C\beta^k{}_l{}^i &= \delta^{ki}\ \delta_l{}^i\ (\upsilon_5)^i \\
{}_R\omega^i{}_j &= \delta^i{}_j\ (\upsilon_7)^i
\end{aligned}
$$

defining vector terms $\upsilon_{4,i}, \upsilon_{5,i}, \upsilon_{7,i}$ implicitly. Remaining terms are

$$
\begin{aligned}
&\upsilon_1 = {}_R\alpha && \upsilon_{2,i} = {}_B\alpha^i && \upsilon_{3,i} = {}_R\beta^i && \upsilon_{6,i} = \operatorname{diag}(\ell_1)^i \\
&\upsilon_{8,i} = \upsilon_{2,i} + \frac{\varpi_{1,i}}{2} && \upsilon_{9,i} = \upsilon_{4,i} - \frac{\varpi_{2,i}}{2} && (\ell_1)^k{}_l = {}_C\alpha^k{}_l && (\ell_2)^k{}_l = {}_D\alpha^k{}_l \\
&(\varpi_1)_i = {}_D\beta_{iii} && (\varpi_2)_i = {}_D\omega_{iiii} && \tilde{\upsilon}_7 = \mathbb{1}_{\upsilon_7 \neq 0\ \vee\ \tilde{c}(0)\neq 0} &&
\end{aligned}
$$

The following additional restrictions cover $\tilde{c}(\tau)_k \neq 0$ to diagonalize interactions with ${}_C\alpha^k{}_l$

$$
(\tilde{\upsilon}_7)^i\ (\tilde{\upsilon}_7)_j\ (\ell_1)^i{}_j = \delta^i{}_j\ (\tilde{\upsilon}_7)^i\ (\upsilon_6)^i
$$

make the k,l traced diagonals of ${}_D\beta^k{}_l{}^i$, ${}_D\omega^k{}_l{}^i{}_j$ diagonal with respect to i,j

$$
\begin{aligned}
(\tilde{\upsilon}_7)^k\ (\tilde{\upsilon}_7)^i\ {}_D\beta^{kki} &= \delta^{ki}\ (\tilde{\upsilon}_7)^i\ (\varpi_1)^i \\
(\tilde{\upsilon}_7)^k\ (\tilde{\upsilon}_7)^i\ (\tilde{\upsilon}_7)_j\ {}_D\omega^{kki}{}_j &= \delta^{ki}\ \delta_j{}^i\ (\tilde{\upsilon}_7)^i\ (\varpi_2)^i
\end{aligned}
$$

and ensure constraint (A.11)

$$
(\upsilon_5)^i\ (\tilde{\upsilon}_7)^i = \mathbb{0}^i\,.
$$

Together, these restrictions diagonalize $c(\tau) = \operatorname{diag}\left((\tilde{c}_1(\tau), \tilde{c}_2(\tau), \ldots)^\intercal\right)$ and yield the diagonalized ODEs above.

A.6 Closed form solutions to diagonalized Riccati ODEs, Lemma 3

Consider the system of ODEs below where υ_1 is a scalar, $\upsilon_2, \ldots, \upsilon_7$ are vectors of length n and ℓ_1, ℓ_2 are matrices of size $n \times n$

$$
\frac{\partial a}{\partial \tau} = \upsilon_1 + \upsilon_2^\intercal b(\tau) + \frac{1}{2}\operatorname{tr}\left(\ell_1 b(\tau) b(\tau)^\intercal\right) + \frac{1}{2}\operatorname{tr}\left(\ell_2 \operatorname{diag}(\tilde{c}(\tau))\right) \tag{A.13}
$$

$$
\begin{aligned}
\frac{\partial b_i}{\partial \tau} =& \upsilon_{3,i} - \upsilon_{4,i} b(\tau)_i + \frac{1}{2}\upsilon_{5,i} b_i^2(\tau) + \upsilon_{8,i}\tilde{c}_i(\tau) + \upsilon_{6,i}\tilde{c}_i(\tau) b_i(\tau) &\text{(A.14)}\\
\frac{\partial \tilde{c}_i}{\partial \tau} =& 2\upsilon_{7,i} - 2\upsilon_{9,i}\tilde{c}_i(\tau) + \upsilon_{6,i}\tilde{c}_i^2(\tau) &\text{(A.15)}
\end{aligned}
$$

with some constant boundaries $a(0)$, $b(0)$ and $\tilde{c}(0)$.

The solution to $a(\tau)$ is

$$
\begin{aligned}
a(\tau) =& a(0) + \upsilon_1 \tau + \upsilon_2^\intercal \int_0^\tau b(s)\,\mathrm{d}s + \frac{1}{2}\operatorname{tr}\left(\ell_1 \int_0^\tau b(s)b(s)^\intercal\,\mathrm{d}s\right)\\
&+ \frac{1}{2}\operatorname{tr}\left(\ell_2 \operatorname{diag}\left(\int_0^\tau \tilde{c}(s)\,\mathrm{d}s\right)\right) \qquad \text{(A.16)}
\end{aligned}
$$

The remaining parts of this section show how to explicitly solve the $b_i(\tau)$ and $\tilde{c}_i(\tau)$ for each i^{th} component under different restrictions. These solutions enter $a(\tau)$ again through the integrals in right hand side of (A.16). Closed form expressions for the integrals of $\tilde{c}_i(\tau)$, $b_i(\tau)$, $b_i^2(\tau)$ and $b_i(\tau)b_j(\tau)$ can be obtained in many cases, for instance through computer algebra systems like Mathematica, but they are not provided here. In my experience, off-diagonal elements $b_i(\tau)b_j(\tau)$ are the most troublesome although their integrals are not necessary when ℓ_1 is a diagonal matrix. When closed form integrals are not available, they can be computed numerically.

Solution when $\{\upsilon_{7,i} = 0$ and $\tilde{c}_i(0) = 0\}$ and $\{\upsilon_{3,i} = 0$ and $b_i(0) = 0\}$ In this case we have that $b(\tau)_i = 0$ and $\tilde{c}_i(\tau) = 0$ thus these components disappear and their associated integrals disappear from (A.16).

Solution when $\{\upsilon_{7,i} = 0$ and $\tilde{c}_i(0) = 0\}$ and $\{\upsilon_{3,i} \neq 0$ or $b_i(0) \neq 0\}$ and $\upsilon_{5,i} = 0$ In this case we have that $\tilde{c}_i(\tau) = 0$ and (A.14) transforms into a simple linear ODE for $b(\tau)_i$ with solution

$$
b_i(\tau) = \begin{cases} b_i(0) + \upsilon_{3,i}\tau & \text{if } \upsilon_{4,i} = 0\\ b_i(0)e^{-\upsilon_{4,i}\tau} + \upsilon_{3,i}\frac{1-e^{-\upsilon_{4,i}\tau}}{\upsilon_{4,i}} & \text{otherwise} \end{cases}.
$$

Solution when $\{\upsilon_{7,i} = 0$ and $\tilde{c}_i(0) = 0\}$ and $\{\upsilon_{3,i} \neq 0$ or $b_i(0) \neq 0\}$ and $\upsilon_{5,i} \neq 0$ In this instance, we have that $\tilde{c}_i(\tau) = 0$ and we focus on b_i. Solutions for $b_i(\tau)$ fall into different cases depending on $\varkappa_i$

$$
\varkappa_i = \upsilon_{4,i}^2 - 2\upsilon_{3,i}\upsilon_{5,i}.
$$

Rearrange (A.14) to integrate over time horizon

$$
\int_0^\tau 1\,\mathrm{d}s = \int_0^\tau \frac{\frac{\partial b_i}{\partial s}}{\upsilon_{3,i} - \upsilon_{4,i} b_i(s) + \frac{1}{2}\upsilon_{5,i} b_i^2(s)}\,\mathrm{d}s
$$

The solution can be found using integration tables

$$b_i(\tau) = \begin{cases} \dfrac{\left(\upsilon_{3,i} + \frac{\left(-\upsilon_{4,i}-\sqrt{\varkappa_i}\right)}{2} b_i(0)\right)\left(1-e^{-\sqrt{\varkappa_i}\tau}\right) + \sqrt{\varkappa_i} b_i(0)}{\sqrt{\varkappa_i} - \frac{\left(\upsilon_{5,i} b_i(0) - \upsilon_{4,i} + \sqrt{\varkappa_i}\right)}{2}\left(1-e^{-\sqrt{\varkappa_i}\tau}\right)} & \text{if } \varkappa_i > 0 \text{ and } \left\{ \begin{array}{c} \sqrt{\varkappa_i} > \upsilon_{5,i} b_i(0) - \upsilon_{4,i} \\ \text{or else} \\ \tau < -\dfrac{\log\left(\frac{\upsilon_{5,i} b_i(0) - \upsilon_{4,i} - \sqrt{\varkappa_i}}{\upsilon_{5,i} b_i(0) - \upsilon_{4,i} + \sqrt{\varkappa_i}}\right)}{\sqrt{\varkappa_i}} \end{array} \right\} \\ \dfrac{\upsilon_{4,i} - \frac{\upsilon_{4,i} - \upsilon_{5,i} b_i(0)}{1 + \frac{\upsilon_{4,i} - \upsilon_{5,i} b_i(0)}{2}\tau}}{\upsilon_{5,i}} & \text{if } \varkappa_i = 0 \text{ and } \left\{ \begin{array}{l} 0 \leq \upsilon_{4,i} - \upsilon_{5,i} b_i(0) \\ \text{or else} \\ \tau < -\dfrac{2}{\upsilon_{4,i} - \upsilon_{5,i} b_i(0)} \end{array} \right\} \\ \dfrac{\upsilon_{4,i} + \sqrt{-\varkappa_i} \tan\left(\arctan\left(\frac{\upsilon_{5,i} b_i(0) - \upsilon_{4,i}}{\sqrt{-\varkappa_i}}\right) + \frac{\sqrt{-\varkappa_i}}{2}\tau\right)}{\upsilon_{5,i}} & \text{if } \left\{ \begin{array}{l} \varkappa_i < 0 \\ \text{and } \tau < \dfrac{\pi - 2\arctan\left(\frac{\upsilon_{5,i} b_i(0) - \upsilon_{4,i}}{\sqrt{-\varkappa_i}}\right)}{\sqrt{-\varkappa_i}} \end{array} \right\} \end{cases} \tag{A.17}$$

Note that π refers to the trigonometric constant in this context.

Solution when $\{\upsilon_{7,i} \neq 0$ **or** $\tilde{c}_i(0) \neq 0\}$ **and** $\upsilon_{5,i} = 0$ In this instance, solutions for $\tilde{c}_i(\tau)$ fall into different cases depending on $\hbar_i$ as detailed by Kim and Omberg (1996)

$$\hbar_i = 4\upsilon_{9,i}^2 - 8\upsilon_{6,i}\upsilon_{7,i}.$$

First we rearrange (A.15) and integrate over time horizon

$$\int_0^\tau 1 \, \mathrm{d}s = \int_0^\tau \frac{\frac{\partial \tilde{c}_i}{\partial s}}{2\upsilon_{7,i} - 2\upsilon_{9,i}\tilde{c}_i(s) + \upsilon_{6,i}\tilde{c}_i^2(s)} \, \mathrm{d}s$$

then we can find the solution using integration tables

$$\tilde{c}_i(\tau) = \begin{cases} 2\upsilon_{7,i}\tau + \tilde{c}_i(0) & \text{if } \upsilon_{6,i} = 0 \text{ and } \upsilon_{9,i} = 0 \\ \upsilon_{7,i}\frac{1-e^{-2\upsilon_{9,i}\tau}}{\upsilon_{9,i}} + \tilde{c}_i(0)e^{-2\upsilon_{9,i}\tau} & \text{if } \upsilon_{6,i} = 0 \text{ and } \upsilon_{9,i} \neq 0 \\ \dfrac{\left(2\upsilon_{7,i} + \left(-\upsilon_{9,i} - \frac{\sqrt{\hbar_i}}{2}\right)\tilde{c}_i(0)\right)\left(1-e^{-\sqrt{\hbar_i}\tau}\right) + \sqrt{\hbar_i}\tilde{c}_i(0)}{\sqrt{\hbar_i} - \left(\upsilon_{6,i}\tilde{c}_i(0) - \upsilon_{9,i} + \frac{\sqrt{\hbar_i}}{2}\right)\left(1-e^{-\sqrt{\hbar_i}\tau}\right)} & \text{if } \left\{ \begin{array}{l} \upsilon_{6,i} \neq 0 \text{ and } \hbar_i > 0 \\ \text{and } \left\{ \begin{array}{c} \upsilon_{9,i} > \upsilon_{6,i}\tilde{c}_i(0) - \dfrac{1}{2}\sqrt{\hbar_i} \\ \text{or else} \\ \tau < -\dfrac{\log\left(\frac{\upsilon_{6,i}\tilde{c}_i(0) - \upsilon_{9,i} - \frac{\sqrt{\hbar_i}}{2}}{\upsilon_{6,i}\tilde{c}_i(0) - \upsilon_{9,i} + \frac{\sqrt{\hbar_i}}{2}}\right)}{\sqrt{\hbar_i}} \end{array} \right\} \end{array} \right\} \\ \dfrac{\upsilon_{9,i} - \frac{\upsilon_{9,i} - \upsilon_{6,i} c_i(0)}{1 + \left(\upsilon_{9,i} - \upsilon_{6,i} c_i(0)\right)\tau}}{\upsilon_{6,i}} & \text{if } \left\{ \begin{array}{l} \upsilon_{6,i} \neq 0 \text{ and } \hbar_i = 0 \\ \text{and } \left\{ \begin{array}{l} 0 \leq \upsilon_{9,i} - \upsilon_{6,i} c_i(0) \\ \text{or else} \\ \tau < -\dfrac{1}{\upsilon_{9,i} - \upsilon_{6,i} c_i(0)} \end{array} \right\} \end{array} \right\} \\ \dfrac{\upsilon_{9,i} + \frac{\sqrt{-\hbar_i}}{2} \tan\left(\arctan\left(\frac{2\upsilon_{6,i}\tilde{c}_i(0) - 2\upsilon_{9,i}}{\sqrt{-\hbar_i}}\right) + \frac{\sqrt{-\hbar_i}}{2}\tau\right)}{\upsilon_{6,i}} & \text{if } \left\{ \begin{array}{l} \upsilon_{6,i} \neq 0 \text{ and } \hbar_i < 0 \\ \text{and } \tau < \dfrac{\pi - 2\arctan\left(\frac{2\upsilon_{6,i}\tilde{c}_i(0) - 2\upsilon_{9,i}}{\sqrt{-\hbar_i}}\right)}{\sqrt{-\hbar_i}} \end{array} \right\} \end{cases} \tag{A.18}$$

Once $\tilde{c}_i(\tau)$ is solved, the expression can be plugged into the ODE for $b_i(\tau)$ (A.14). Solving this inhomogenous linear PDE is straightforward as long as it remains finite

$$b_i(\tau) = \frac{\int_0^\tau e^{\int_0^s (\upsilon_{4,i}-\upsilon_{6,i}\tilde{c}_i(u))\mathrm{d}u} \left(\upsilon_{3,i}+\upsilon_{8,i}\tilde{c}_i(s)\right)\mathrm{d}s + b_i(0)}{e^{\int_0^\tau (\upsilon_{4,i}-\upsilon_{6,i}\tilde{c}_i(s))\mathrm{d}s}}.$$

Known closed form solutions to $b_i(\tau)$ are displayed separately for the main groups of cases depending on the values of $\upsilon_{6,i}$ and $\hbar_i$.

If $\upsilon_{6,i} = 0$

$$b_i(\tau) = \begin{cases} b_i(0) + \left(\upsilon_{3,i}+\upsilon_{8,i}\tilde{c}_i(0)\right)\tau + \upsilon_{8,i}\upsilon_{7,i}\tau^2 & \text{if } \upsilon_{9,i}=0 \text{ and } \upsilon_{4,i}=0 \\ b_i(0)e^{-\upsilon_{4,i}\tau} + 2\frac{\upsilon_{8,i}\upsilon_{7,i}}{\upsilon_{4,i}}\tau + \left(\upsilon_{3,i}+\upsilon_{8,i}\left(\tilde{c}_i(0)-2\frac{\upsilon_{7,i}}{\upsilon_{4,i}}\right)\right)\frac{1-e^{-\upsilon_{4,i}\tau}}{\upsilon_{4,i}} & \text{if } \upsilon_{9,i}=0 \text{ and } \upsilon_{4,i}\neq 0 \\ b_i(0) + \left(\upsilon_{3,i}+\frac{\upsilon_{8,i}\upsilon_{7,i}}{\upsilon_{9,i}}\right)\tau + \upsilon_{8,i}\left(\tilde{c}_i(0)-\frac{\upsilon_{7,i}}{\upsilon_{9,i}}\right)\frac{1-e^{-2\upsilon_{9,i}\tau}}{2\upsilon_{9,i}} & \text{if } \upsilon_{9,i}\neq 0 \text{ and } \upsilon_{4,i}=0 \\ \begin{pmatrix} b_i(0)e^{-\upsilon_{4,i}\tau} + \left(\upsilon_{3,i}+2\frac{\upsilon_{8,i}\upsilon_{7,i}}{\upsilon_{4,i}}\right)\frac{1-e^{-\upsilon_{4,i}\tau}}{\upsilon_{4,i}} \\ +\,\upsilon_{8,i}\left(\tilde{c}_i(0)-\frac{2\upsilon_{7,i}}{\upsilon_{4,i}}\right)\tau e^{-\upsilon_{4,i}\tau} \end{pmatrix} & \text{if } 2\upsilon_{9,i}=\upsilon_{4,i}\neq 0 \\ \begin{pmatrix} b_i(0)e^{-\upsilon_{4,i}\tau} + \left(\upsilon_{3,i}+\frac{\upsilon_{8,i}\upsilon_{7,i}}{\upsilon_{9,i}}\right)\frac{1-e^{-\upsilon_{4,i}\tau}}{\upsilon_{4,i}} \\ +\,\upsilon_{8,i}\left(\frac{\upsilon_{7,i}}{\upsilon_{9,i}}-\tilde{c}_i(0)\right)\frac{e^{-2\upsilon_{9,i}\tau}-e^{-\upsilon_{4,i}\tau}}{2\upsilon_{9,i}-\upsilon_{4,i}} \end{pmatrix} & \text{if } \begin{Bmatrix} \upsilon_{9,i}\neq 0 \text{ and } \upsilon_{4,i}\neq 0 \\ \text{and } 2\upsilon_{9,i}\neq\upsilon_{4,i} \end{Bmatrix} \end{cases}$$

If $\upsilon_{6,i} \neq 0$ and $\hbar_i > 0$ and $\tilde{c}_i(0) = 0$

$$b_i(\tau) = \begin{cases} 2\upsilon_{7,i}\frac{\begin{pmatrix} \frac{\sqrt{\hbar_i}}{2\upsilon_{7,i}}b_i(0) \\ +\left(\upsilon_{8,i}+\left(\upsilon_{9,i}+\frac{\sqrt{\hbar_i}}{2}\right)\frac{\upsilon_{3,i}}{2\upsilon_{7,i}}\right)\tau \\ -\left(\upsilon_{8,i}+\left(\upsilon_{9,i}-\frac{\sqrt{\hbar_i}}{2}\right)\frac{\upsilon_{3,i}}{2\upsilon_{7,i}}\right)\frac{1-e^{-\sqrt{\hbar_i}\tau}}{\sqrt{\hbar_i}} \end{pmatrix}}{\sqrt{\hbar_i}-\left(-\upsilon_{9,i}+\frac{1}{2}\sqrt{\hbar_i}\right)(1-e^{-\sqrt{\hbar_i}\tau})} & \text{if } \upsilon_{9,i}-\upsilon_{4,i}=\frac{\sqrt{\hbar_i}}{2} \\ 2\upsilon_{7,i}\frac{\begin{pmatrix} \frac{\sqrt{\hbar_i}}{2\upsilon_{7,i}}b_i(0)e^{-\sqrt{\hbar_i}\tau} \\ +\left(\upsilon_{8,i}+\left(\upsilon_{9,i}+\frac{\sqrt{\hbar_i}}{2}\right)\frac{\upsilon_{3,i}}{2\upsilon_{7,i}}\right)\frac{1-e^{-\sqrt{\hbar_i}\tau}}{\sqrt{\hbar_i}} \\ -\left(\upsilon_{8,i}+\left(\upsilon_{9,i}-\frac{\sqrt{\hbar_i}}{2}\right)\frac{\upsilon_{3,i}}{2\upsilon_{7,i}}\right)\tau e^{-\sqrt{\hbar_i}\tau} \end{pmatrix}}{\sqrt{\hbar_i}-\left(-\upsilon_{9,i}+\frac{1}{2}\sqrt{\hbar_i}\right)(1-e^{-\sqrt{\hbar_i}\tau})} & \text{if } \upsilon_{9,i}-\upsilon_{4,i}=-\frac{\sqrt{\hbar_i}}{2} \\ 2\upsilon_{7,i}\frac{\left(K_{1,i}+K_{2,i}e^{-\sqrt{\hbar_i}\tau}+\left(b_i(0)\frac{\sqrt{\hbar_i}}{2\upsilon_{7,i}}-K_{1,i}-K_{2,i}\right)e^{\left(\upsilon_{9,i}-\upsilon_{4,i}-\frac{\sqrt{\hbar_i}}{2}\right)\tau}\right)}{\sqrt{\hbar_i}-\left(-\upsilon_{9,i}+\frac{\sqrt{\hbar_i}}{2}\right)(1-e^{-\sqrt{\hbar_i}\tau})} & \text{if } |\upsilon_{9,i}-\upsilon_{4,i}|\neq\frac{\sqrt{\hbar_i}}{2} \end{cases}$$

under the restrictions

$$\upsilon_{9,i} > -\frac{\sqrt{\hbar_i}}{2} \qquad \text{or else} \qquad \tau < -\frac{\log\left(\frac{-2\upsilon_{9,i}-\sqrt{\hbar_i}}{-2\upsilon_{9,i}+\sqrt{\hbar_i}}\right)}{\sqrt{\hbar_i}}$$

where

$$K_{1,i} = \frac{2\upsilon_{8,i} + \left(\upsilon_{9,i} + \frac{\sqrt{\hbar_i}}{2}\right)\frac{\upsilon_{3,i}}{\upsilon_{7,i}}}{\sqrt{\hbar_i} - 2\left(\upsilon_{9,i} - \upsilon_{4,i}\right)}$$

$$K_{2,i} = \frac{2\upsilon_{8,i} + \left(\upsilon_{9,i} - \frac{\sqrt{\hbar_i}}{2}\right)\frac{\upsilon_{3,i}}{\upsilon_{7,i}}}{\sqrt{\hbar_i} + 2\left(\upsilon_{9,i} - \upsilon_{4,i}\right)}.$$

If $\upsilon_{6,i} \neq 0$ and $\hbar_i = 0$

$$b_i(\tau) = \begin{cases} \frac{b_i(0) + \left(\upsilon_{3,i} + \upsilon_{8,i}\frac{\upsilon_{4,i}}{\upsilon_{6,i}}\right)\left(\tau + \frac{\upsilon_{4,i}}{2}\tau^2\right)}{1+\upsilon_{4,i}\tau} & \text{if } \left\{\begin{matrix} \upsilon_{9,i} = \upsilon_{4,i} \\ \text{and } \upsilon_{9,i} = \upsilon_{6,i}c_i(0) \end{matrix}\right\} \\ \frac{\left(\begin{matrix} b_i(0) + \left(\upsilon_{3,i} + \upsilon_{8,i}\frac{\upsilon_{4,i}}{\upsilon_{6,i}}\right)\left(\tau + \frac{\upsilon_{4,i}}{2}\tau^2\right) \\ - \frac{\upsilon_{8,i}}{\upsilon_{6,i}}\left(\upsilon_{4,i} - \upsilon_{6,i}c_i(0)\right)\tau \\ - c_i(0)\upsilon_{8,i}\left(\tau - \frac{\log(1 + \left(\upsilon_{4,i} - \upsilon_{6,i}c_i(0)\right)\tau)}{\upsilon_{4,i} - \upsilon_{6,i}c_i(0)}\right) \end{matrix}\right)}{1+\upsilon_{4,i}\tau} & \text{if } \left\{\begin{matrix} \upsilon_{9,i} = \upsilon_{4,i} \\ \text{and } \upsilon_{9,i} \neq \upsilon_{6,i}c_i(0) \end{matrix}\right\} \\ \frac{\left(\begin{matrix} b_i(0) \\ + \left(\upsilon_{3,i} + \frac{\upsilon_{8,i}}{\upsilon_{6,i}}\upsilon_{9,i}\right)\left(\begin{matrix} \frac{1 - e^{-(\upsilon_{9,i} - \upsilon_{4,i})\tau}}{\upsilon_{9,i} - \upsilon_{4,i}} \\ - \upsilon_{9,i}\frac{\tau e^{-(\upsilon_{9,i} - \upsilon_{4,i})\tau} - \frac{1 - e^{-(\upsilon_{9,i} - \upsilon_{4,i})\tau}}{\upsilon_{9,i} - \upsilon_{4,i}}}{\upsilon_{9,i} - \upsilon_{4,i}} \end{matrix}\right) \end{matrix}\right)}{(1+\upsilon_{9,i}\tau)e^{-(\upsilon_{9,i} - \upsilon_{4,i})\tau}} & \text{if } \left\{\begin{matrix} \upsilon_{9,i} \neq \upsilon_{4,i} \\ \text{and } \upsilon_{9,i} = \upsilon_{6,i}c_i(0) \end{matrix}\right\} \\ \frac{\left(\begin{matrix} b_i(0) + \upsilon_{3,i}\frac{1 - e^{-(\upsilon_{9,i} - \upsilon_{4,i})\tau}}{\upsilon_{9,i} - \upsilon_{4,i}} \\ - \frac{\left(\upsilon_{3,i} + \upsilon_{8,i}\frac{\upsilon_{9,i}}{\upsilon_{6,i}}\right)\upsilon_{9,i}}{\upsilon_{9,i} - \upsilon_{4,i}}\left(\tau e^{-(\upsilon_{9,i} - \upsilon_{4,i})\tau} - \frac{1 - e^{-(\upsilon_{9,i} - \upsilon_{4,i})\tau}}{\upsilon_{9,i} - \upsilon_{4,i}}\right) \\ + c_i(0)\upsilon_{8,i}e^{\frac{\upsilon_{9,i} - \upsilon_{4,i}}{\upsilon_{9,i} - \upsilon_{6,i}c_i(0)}} \\ \cdot \frac{\left(\begin{matrix} \mathrm{Ei}\left(\frac{-(\upsilon_{9,i} - \upsilon_{4,i})}{\upsilon_{9,i} - \upsilon_{6,i}c_i(0)} - (\upsilon_{9,i} - \upsilon_{4,i})\tau\right) \\ - \mathrm{Ei}\left(\frac{-(\upsilon_{9,i} - \upsilon_{4,i})}{\upsilon_{9,i} - \upsilon_{6,i}c_i(0)}\right) \end{matrix}\right)}{\upsilon_{9,i} - \upsilon_{6,i}c_i(0)} \end{matrix}\right)}{(1+\upsilon_{9,i}\tau)e^{-(\upsilon_{9,i} - \upsilon_{4,i})\tau}} & \text{if } \left\{\begin{matrix} \upsilon_{9,i} \neq \upsilon_{4,i} \\ \text{and } \upsilon_{9,i} \neq \upsilon_{6,i}c_i(0) \end{matrix}\right\} \end{cases}$$

under the restrictions

$$\upsilon_{9,i} \geq 0 \qquad \text{or else} \qquad \tau < -\frac{1}{\upsilon_{9,i}}.$$

where Ei is the exponential integral Ei.

If $\upsilon_{6,i} \neq 0$ and $\hbar_i < 0$

$$b_i(\tau) = \left(\begin{array}{l} b_i(0) e^{-(\upsilon_{4,i}-\upsilon_{9,i})\tau} \sec\left(K_{3,i} + \frac{\sqrt{-\hbar_i}}{2}\tau\right) \cos\left(K_{3,i}\right) \\ + \frac{\left(\begin{array}{l} K_{4,i} + K_{5,i} \tan\left(K_{3,i} + \frac{\sqrt{-\hbar_i}}{2}\tau\right) \\ - e^{-(\upsilon_{4,i}-\upsilon_{9,i})\tau} \sec\left(K_{3,i} + \frac{\sqrt{-\hbar_i}}{2}\tau\right) \left(\begin{array}{l} K_{4,i} \cos\left(K_{3,i}\right) \\ + K_{5,i} \sin\left(K_{3,i}\right) \end{array} \right) \end{array} \right)}{\left(\upsilon_{4,i} - \upsilon_{9,i}\right)^2 - \frac{\hbar_i}{4}} \end{array} \right)$$

under the restriction

$$\tau < \frac{\pi - 2\arctan\left(\frac{2\upsilon_{6,i}\tilde{c}_i(0) - 2\upsilon_{9,i}}{\sqrt{-\hbar_i}}\right)}{\sqrt{-\hbar_i}}$$

where

$$\begin{aligned} K_{3,i} =& \arctan\left(\frac{2\upsilon_{6,i}\tilde{c}_i(0) - 2\upsilon_{9,i}}{\sqrt{-\hbar_i}}\right) \\ K_{4,i} =& \upsilon_{3,i}\left(\upsilon_{4,i} - \upsilon_{9,i}\right) + \upsilon_{8,i}\left(\frac{\upsilon_{9,i}\upsilon_{4,i}}{\upsilon_{6,i}} - 2\upsilon_{7,i}\right) \\ K_{5,i} =& \left(\upsilon_{3,i} + \frac{\upsilon_{8,i}}{\upsilon_{6,i}}\upsilon_{4,i}\right)\frac{\sqrt{-\hbar_i}}{2}. \end{aligned}$$

Closed form expressions for integrals of $\tilde{c}_i$, b_i and b_i^2 can be obtained for many cases. Imposing restrictions like $\upsilon_{9,i} = \upsilon_{4,i}$ and $\tilde{c}_i(0) = b_i(0) = 0$ helps to find closed form expressions for tough cases.

Solution when $\{\upsilon_{7,i} \neq 0$ **or** $\tilde{c}_i(0) \neq 0\}$ **and** $\{\upsilon_{3,i} = 0$ **and** $b_i(0) = 0\}$ **and** $\upsilon_{8,i} = 0$ Then $b(\tau)_i = 0$ and $\tilde{c}(\tau)_i$ coincides with (A.18).

Solution when $\{\upsilon_{7,i} \neq 0$ **or** $\tilde{c}_i(0) \neq 0\}$ **and** $\upsilon_{5,i} \neq 0$ **and** $\upsilon_{6,i} = 0$ **and** $\upsilon_{8,i} = 0$ Then $b(\tau)_i$ and $\tilde{c}(\tau)_i$ coincide with restricted instances of (A.17) and (A.18). Although they technically solve (A.14) (A.15), these constraints do not correspond to quadratic cases in Section A.5.

A.7 Proof of generic PDE separation, Lemma 4

Substituting the Ansatz (16) and coefficients (15) into (14), it is easy to verify that functions $g_i(t, X_i)$ satisfying (17) and boundary conditions $g_i(T, X_i)$ also solve $g(t, X)$.

More generally, suppose that the antidiagonal of C is not zero

$$C = \begin{pmatrix} C_1 & C_0 \\ C_0^\intercal & C_2 \end{pmatrix},$$

and that the boundary condition admits the decomposition$g(T,X) = g_0(T)g_1(T,X_1)g_2(T,X_2)$. Using the Ansatz

$$g(t,X) = g_0(t)g_1(t,X_1)g_2(t,X_2)$$

and substituting into (14) we arrive at

$$\begin{aligned}
0 =& g_1(t,X_1)g_2(t,X_2)\left(\frac{\partial g_0}{\partial t} + g_0(t)\left(\frac{\partial \log g_1}{\partial X_1}\right)^{\intercal} C_0 \frac{\partial \log g_2}{\partial X_2}\right) \\
&+ g_0(t)g_2(t,X_2)\left(\frac{\partial g_1}{\partial t} + g_1(t,X_1)R_1 + g_{1,X_1}^{\intercal} B_1 + \frac{1}{2}\frac{g_{1,X_1}^{\intercal}\left(C_1 - D_1\right) g_{1,X_1}}{g_1(t,X_1)} + \frac{1}{2}\operatorname{tr}\left(g_{1,X_1X_1^{\intercal}} D_1\right)\right) \\
&+ g_0(t)g_1(t,X_1)\left(\frac{\partial g_2}{\partial t} + g_2(t,X_2)R_2 + g_{2,X_2}^{\intercal} B_2 + \frac{1}{2}\frac{g_{2,X_2}^{\intercal}\left(C_2 - D_2\right) g_{2,X_2}}{g_2(t,X_2)} + \frac{1}{2}\operatorname{tr}\left(g_{2,X_2X_2^{\intercal}} D_2\right)\right)
\end{aligned}$$

We can readily verify that PDE above is zero when the PDEs (17) hold and either $C_0 = 0$ and $g_0(t) = g_0(T)$; or C_0 does not depend on X and g_1, g_2 are log-linear on their respective X_i, that is, their space derivatives of their logarithms don't depend on X.

$$0 = \frac{\partial g_0}{\partial t} + g_0(t)\left(\frac{\partial \log g_1}{\partial X_1}\right)^{\intercal} C_0 \frac{\partial \log g_2}{\partial X_2}$$

Under such conditions, the solution to the ODE above with boundary condition $g_0(T)$ is

$$g_0(t) = g_0(T)e^{\int_t^T \left(\frac{\partial \log g_1}{\partial X_1}\right)^{\intercal} C_0 \frac{\partial \log g_2}{\partial X_2} \mathrm{d}s}.$$

A.8 Proof of price for payoff stream, Lemma 5

The price $\Upsilon(t, X_t, Q_t; T)$ of the claim to a payoff stream Q_t from present time t up to date T given the pricing kernel K_t with market price of risk Λ corresponds to the price of a portfolio composed by terminal payoff claims that provides an equivalent payoff stream. By Lemma 2 we have that

$$\Upsilon(t, X_t, Q_t; T) = E\left[\int_t^T \frac{K_u}{K_t} Q_u \,\mathrm{d}u \,|\mathcal{F}_t\right] = \int_t^T \Omega(t, X_t, Q_t; u)\,\mathrm{d}u = Q_t \int_t^T \tilde{\Omega}(t, X_t; u)\,\mathrm{d}u\,. \tag{A.19}$$

A.9 Proof of replicating strategy for payoff stream, Lemma 5

Suppose that markets are complete and we want to find the minimum initial capital $\Upsilon(t, X_t, Q_t; T)$ necessary to replicate a payoff stream Q_t and make sure that at the end the value of our portfolio is zero $\Upsilon(T, X_T, Q_T; T) = 0$. Changes in the value of the replicating portfolio reflect the payouts Q_t and the proceeds from investing funds according to the investment strategy π_t

$$\mathrm{d}\Upsilon_t = -Q_t\,\mathrm{d}t + \Upsilon(t, X_t, Q_t; T)\left(\pi_t^{\intercal}(\mu_A - r\mathbb{1}) + r\right)\mathrm{d}t + \Upsilon(t, X_t, Q_t; T)\pi_t^{\intercal}\Sigma_A\,\mathrm{d}Z_{A,t} \tag{A.20}$$

Assuming that X_t and Q_t are tradeable through A_t and applying Itô's lemma to $\Upsilon(t, X_t, Q_t)$ shows that

$$\mathrm{d}\Upsilon_t = \frac{\partial \Upsilon}{\partial t}\,\mathrm{d}t + \Upsilon_X^{\intercal}\left(\mu_X\,\mathrm{d}t + \Sigma_X\rho_{XA}\,\mathrm{d}Z_{A,t}\right) + \Upsilon_Q Q_t\left(\mu_Q\,\mathrm{d}t + \Sigma_Q\rho_{QA}\,\mathrm{d}Z_{A,t}\right)$$

$$+\frac{1}{2}\operatorname{tr}\left(\Upsilon_{XX^\intercal}\Sigma_X\Sigma_X^\intercal\right)\mathrm{d}t+\frac{1}{2}Q_t^2\Upsilon_{QQ}\Sigma_Q\Sigma_Q^\intercal\,\mathrm{d}t+Q_t\Upsilon_{XQ}^\intercal\Sigma_X\rho_{XQ}\Sigma_Q^\intercal\,\mathrm{d}t \tag{A.21}$$

There is a unique strategy π_t that deterministically replicates the desired dynamics by cancelling out the Brownian motions on both sides (A.20) and (A.21) of $\mathrm{d}\Upsilon_t$

$$\pi_t=(\Sigma_A^\intercal)^{-1}\,\frac{\rho_{XA}^\intercal\Sigma_X^\intercal\Upsilon_X+\rho_{QA}^\intercal\Sigma_Q^\intercal Q_t\Upsilon_Q}{\Upsilon(t,X_t,Q_t;T)}$$

Using this investment strategy, the required initial capital $\Upsilon(t,X_t,Q_t;T)$ is the solution to the following PDE

$$\begin{aligned}0=&Q_t-\Upsilon(t,X_t,Q_t;T)r+\Upsilon_X^\intercal\left(\mu_X-\Sigma_X\rho_{XA}\Sigma_A^{-1}(\mu_A-r\mathbb{1})\right)\\&+Q_t\Upsilon_Q\left(\mu_Q-\Sigma_Q\rho_{QA}\Sigma_A^{-1}(\mu_A-r\mathbb{1})\right)\\&+\frac{\partial\Upsilon}{\partial t}+\frac{1}{2}\operatorname{tr}\left(\Upsilon_{XX^\intercal}\Sigma_X\Sigma_X^\intercal\right)+\frac{1}{2}\Upsilon_{QQ}Q_t\Sigma_Q\Sigma_Q^\intercal Q_t+\Upsilon_{XQ}^\intercal\Sigma_X\rho_{XQ}\Sigma_Q^\intercal Q_t\end{aligned} \tag{A.22}$$

with boundary condition $\Upsilon(T,X_T,Q_T;T)=0$.

It can be readily verified that (A.19) is the solution to this PDE

$$\Upsilon(t,X_t,Q_t;T)=\int_t^T\Omega(t,X_t,Q_t;s)\,\mathrm{d}s$$

obtaining

$$\begin{aligned}0=&\overbrace{Q_t}^{\Omega(t,X_t,Q_t;t)}-\Omega(t,X_t,Q_t;t)\\&+\int_t^T\underbrace{\begin{pmatrix}\frac{\partial\Omega}{\partial t}-\Omega(t,X_t,Q_t;s)r\\+\Omega_X^\intercal\left(\mu_X-\Sigma_X\rho_{XA}\Sigma_A^{-1}(\mu_A-r\mathbb{1})\right)\\+Q_t\Omega_Q\left(\mu_Q-\Sigma_Q\rho_{QA}\Sigma_A^{-1}(\mu_A-r\mathbb{1})\right)\\+\frac{1}{2}\operatorname{tr}\left(\Omega_{XX^\intercal}\Sigma_X\Sigma_X^\intercal\right)+\frac{1}{2}\Omega_{QQ}Q_t\Sigma_Q\Sigma_Q^\intercal Q_t+\Omega_{XQ}^\intercal\Sigma_X\rho_{XQ}\Sigma_Q^\intercal Q_t\end{pmatrix}}_{0}\mathrm{d}s\end{aligned}$$

and noticing that the integrand corresponds to the zero condition of the PDE in (A.4) for Ω.

A.10 Proof of stationary house price to income ratio, Lemma 6

Stationarity here is understood in the sense that the unconditional joint distribution of process elements is independent of time, that is, the distribution is invariant to time shifts.

The discounted future payoff price to current payoff ratio is

$$\frac{\Omega(t,X_t,Q_t;T)}{Q_t}=\tilde{\Omega}(t,X_t;T) \tag{A.23}$$

and the payoff stream price to payoff ratio is

$$\frac{\Upsilon(t,X_t,Q_t;T)}{Q_t}=\int_t^T\tilde{\Omega}(t,X_t;s)\,\mathrm{d}s\,. \tag{A.24}$$

Observe that $\tilde{\Omega}$ dynamics described in (13) and its associated terminal value depend only on market parameters. Assuming that market parameters are Markovian with respect to state process X_t, then $\tilde{\Omega}$ dynamics only depend on X_t. In this setting, the only time references used to describe $\tilde{\Omega}$ are the initial t and terminal s times of the process. Neither dynamics nor boundary values depend on time; the initial and terminal time points only affect the time horizon over which $\tilde{\Omega}$ is defined. The PDE characterizing $\tilde{\Omega}$ has deterministic parameters and describes a deterministic function of inputs $s-t$ and X_t. The expression in (A.23) is equivalent to

$$\hat{\Omega}(T-t, X_t) = \tilde{\Omega}(t, X_t; T)$$

and the integral (A.24) can be reformulated in terms of $\hat{\Omega}(s-t, X_t) = \tilde{\Omega}(t, X_t; s)$

$$\hat{\Upsilon}(T-t, X_t) = \int_0^{T-t} \hat{\Omega}(h, X_t)\, \mathrm{d}h$$

to obtain functions of inputs $T-t$ and X_t. Keeping the time horizon $T-t$ constant in the above expressions through partial function application yields a mapping function that only depends on X_t. As X_t is assumed to be jointly stationary, this makes the mapped process stationary as well. A similar argument applies when assuming that the reference horizon $T-t$ is jointly stationary with state. Note that so far I implicitly assumed that the mapping function is measurable since this condition makes it possible to obtain a well-defined pushforward measure (Kallenberg, 2021, Lemma 25.1) in this context.

The price-to-rent ratio $\frac{P_H}{N_t}$ can be expressed through the equivalence (18) as an instance of (A.24) with $P_H = \Upsilon(t, X_t, Q_t; T)$ and $N_t = Q_t$, so the stationarity arguments applies verbatim

$$\frac{P_H}{N_t} = \frac{\Upsilon(t, X_t, Q_t; T)}{Q_t} = \int_t^T \tilde{\Omega}(t, X_t; s)\, \mathrm{d}s\,. \tag{A.25}$$

Regarding the price-to-income ratio,

$$\frac{P_H}{L_t} = \frac{P_H}{N_t} e^{\nu_{N,t} - \nu_{L,t}}$$

it is also a deterministic function of the time horizon $T-t$ and the state process X_t since both $\frac{P_H}{N_t}$ and $e^{\nu_{N,t}-\nu_{L,t}}$ are as well. Notice that $\nu_{N,t}$ and $\nu_{L,t}$ are both components of the state process X_t. At this point, the arguments above become applicable here.

A.11 Proof of payoff claims returns, Lemma 7

The dynamics of the terminal payoff claim investment $A_{\Omega,t}$ in (23) can be derived from (20) by expanding definitions and applying Itô's lemma

$$\begin{aligned}
\frac{\mathrm{d}A_{\Omega,t}}{A_{\Omega,t}} =& \frac{\mathrm{d}\Omega(t, X_t, Q_t)}{\Omega(t, X_t, Q_t)} \\
=& \frac{\mathrm{d}Q_t}{Q_t} + \frac{\mathrm{d}\tilde{\Omega}(t, X_t)}{\tilde{\Omega}(t, X_t)} + \frac{\tilde{\Omega}_X^\intercal \Sigma_X \rho_{XQ} \Sigma_Q^\intercal}{\tilde{\Omega}(t, X_t)}\, \mathrm{d}t \\
=& \left(\mu_Q + \frac{\frac{\partial \tilde{\Omega}}{\partial t} + \tilde{\Omega}_X^\intercal \left(\mu_X + \Sigma_X \rho_{XQ} \Sigma_Q^\intercal\right) + \frac{1}{2} \operatorname{tr}\left(\tilde{\Omega}_{XX^\intercal} \Sigma_X \Sigma_X^\intercal\right)}{\tilde{\Omega}(t, X_t)} \right) \mathrm{d}t
\end{aligned}$$

$$+ \Sigma_Q \, \mathrm{d}Z_{Q,t} + \frac{\tilde{\Omega}_X^\intercal \Sigma_X}{\tilde{\Omega}(t, X_t)} \, \mathrm{d}Z_{X,t}$$

At this point I replace $\frac{\partial \tilde{\Omega}}{\partial t}$ with the value implicitly defined by the PDE (13) and after some simplifications we arrive at

$$\frac{\mathrm{d}A_{\Omega,t}}{A_{\Omega,t}} = r \, \mathrm{d}t + \Sigma_Q \left(\rho_{QA} \Lambda \, \mathrm{d}t + \mathrm{d}Z_{Q,t} \right) + \frac{\tilde{\Omega}_X^\intercal}{\tilde{\Omega}(t, X_t)} \Sigma_X \left(\rho_{XA} \Lambda \, \mathrm{d}t + \mathrm{d}Z_{X,t} \right).$$

where Λ is the market price of risk from (1). Further substituting $\mathrm{d}Z_{Q,t}$ by $\rho_{QA} \, \mathrm{d}Z_{A,t}$ and $\mathrm{d}Z_{X,t}$ by $\rho_{XA} \, \mathrm{d}Z_{A,t}$ yields

$$\frac{\mathrm{d}A_{\Omega,t}}{A_{\Omega,t}} = r \, \mathrm{d}t + \left(\rho_{QA}^\intercal \Sigma_Q^\intercal + \rho_{XA}^\intercal \Sigma_X^\intercal \frac{\tilde{\Omega}_X}{\tilde{\Omega}(t, X_t)} \right)^\intercal \left(\Lambda \, \mathrm{d}t + \mathrm{d}Z_{A,t} \right).$$

The dynamics of the payoff stream claim investment $A_{\Upsilon,t}$ in (25) can be derived from (22) by expanding definitions and applying Itô's lemma

$$\begin{aligned}
\frac{\mathrm{d}A_{\Upsilon,t}}{A_{\Upsilon,t}} =& \frac{Q_t}{\Upsilon(t, X_t, Q_t)} \, \mathrm{d}t + \frac{\mathrm{d}Q_t}{Q_t} + \frac{\mathrm{d} \int_t^T \tilde{\Omega}(t, X_t; s) \, \mathrm{d}s}{\int_t^T \tilde{\Omega}(t, X_t; s) \, \mathrm{d}s} + \frac{\left(\int_t^T \frac{\partial \tilde{\Omega}(t, X_t; s)}{\partial X} \, \mathrm{d}s \right)^\intercal \Sigma_X \rho_{XQ} \Sigma_Q^\intercal}{\int_t^T \tilde{\Omega}(t, X_t; s) \, \mathrm{d}s} \, \mathrm{d}t \\
=& \frac{\int_t^T \frac{\partial \tilde{\Omega}(t, X_t; s)}{\partial t} \, \mathrm{d}s}{\int_t^T \tilde{\Omega}(t, X_t; s) \, \mathrm{d}s} \, \mathrm{d}t \\
&+ \mu_Q \, \mathrm{d}t + \frac{\left(\int_t^T \frac{\partial \tilde{\Omega}(t, X_t; s)}{\partial X} \, \mathrm{d}s \right)^\intercal \left(\mu_X + \Sigma_X \rho_{XQ} \Sigma_Q^\intercal \right) + \frac{1}{2} \operatorname{tr} \left(\int_t^T \frac{\partial^2 \tilde{\Omega}(t, X_t; s)}{\partial X \partial X^\intercal} \, \mathrm{d}s \, \Sigma_X \Sigma_X^\intercal \right)}{\int_t^T \tilde{\Omega}(t, X_t; s) \, \mathrm{d}s} \, \mathrm{d}t \\
&+ \Sigma_Q \, \mathrm{d}Z_{Q,t} + \frac{\left(\int_t^T \frac{\partial \tilde{\Omega}(t, X_t; s)}{\partial X} \, \mathrm{d}s \right)^\intercal \Sigma_X}{\int_t^T \tilde{\Omega}(t, X_t; s) \, \mathrm{d}s} \, \mathrm{d}Z_{X,t}
\end{aligned}$$

At this point I replace $\frac{\partial \tilde{\Omega}(t, X_t; s)}{\partial t}$ with the value implicitly defined by the PDE (13) and after some simplifications we arrive at

$$\frac{\mathrm{d}A_{\Upsilon,t}}{A_{\Upsilon,t}} = r \, \mathrm{d}t + \Sigma_Q \left(\rho_{QA} \Lambda \, \mathrm{d}t + \mathrm{d}Z_{Q,t} \right) + \frac{\left(\int_t^T \frac{\partial \tilde{\Omega}(t, X_t; s)}{\partial X} \, \mathrm{d}s \right)^\intercal \Sigma_X}{\int_t^T \tilde{\Omega}(t, X_t; s) \, \mathrm{d}s} \left(\rho_{XA} \Lambda \, \mathrm{d}t + \mathrm{d}Z_{X,t} \right).$$

Further substituting $\mathrm{d}Z_{Q,t}$ by $\rho_{QA} \, \mathrm{d}Z_{A,t}$ and $\mathrm{d}Z_{X,t}$ by $\rho_{XA} \, \mathrm{d}Z_{A,t}$ yields

$$\frac{\mathrm{d}A_{\Upsilon,t}}{A_{\Upsilon,t}} = r \, \mathrm{d}t + \left(\rho_{QA}^\intercal \Sigma_Q^\intercal + \rho_{XA}^\intercal \Sigma_X^\intercal \frac{\int_t^T \frac{\partial \tilde{\Omega}(t, X_t; s)}{\partial X} \, \mathrm{d}s}{\int_t^T \tilde{\Omega}(t, X_t; s) \, \mathrm{d}s} \right)^\intercal \left(\Lambda \, \mathrm{d}t + \mathrm{d}Z_{A,t} \right).$$

The explicit expression (27) for the cumulative total return of payoff stream claim $A_{\Upsilon,t}$ can be derived from (22). Applying Itô's lemma, we arrive at

$$\mathrm{d} \log(A_{\Upsilon,t}) = \frac{Q_t}{\Upsilon(t, X_t, Q_t)} \, \mathrm{d}t + \mathrm{d} \log(\Upsilon(t, X_t, Q_t))$$

Integrating the equation above with respect to time yields

$$\frac{A_{\Upsilon,t}}{A_{\Upsilon,t_0}} = \frac{\Upsilon(t, X_t, Q_t)}{\Upsilon(t_0, X_{t_0}, Q_{t_0})} e^{\int_{t_0}^t \frac{Q_s}{\Upsilon(s, X_s, Q_s)} \mathrm{d}s}. \tag{A.26}$$

It is straightforward to apply Itô's lemma and verify that dynamics match (22). Also, one can verify that (A.26) coincides with (21). Using the fundamental theorem of calculus, we can decompose the exponential expression as

$$e^{\int_{t_0}^t \frac{Q_s}{\Upsilon(s,X_s,Q_s)}\mathrm{d}s} = 1 + \int_{t_0}^t \frac{Q_s}{\Upsilon(s,X_s,Q_s)} e^{\int_s^t \frac{Q_u}{\Upsilon(u,X_u,Q_u)}\mathrm{d}u}\,\mathrm{d}s\,.$$

Replacing the expression above in (A.26), as well as using the initial portfolio value equality $A_{\Upsilon,t_0} = \Upsilon(t_0, X_{t_0}, Q_{t_0})$ to cancel out some terms, we arrive at

$$\begin{aligned} A_{\Upsilon,t} =& \Upsilon(t,X_t,Q_t) + \int_{t_0}^t Q_s \frac{\Upsilon(t,X_t,Q_t)}{\Upsilon(s,X_s,Q_s)} e^{\int_s^t \frac{Q_u}{\Upsilon(u,X_u,Q_u)}\mathrm{d}u}\,\mathrm{d}s \\ =& \Upsilon(t,X_t,Q_t) + \int_{t_0}^t Q_s \frac{A_{\Upsilon,t}}{A_{\Upsilon,s}}\,\mathrm{d}s\,. \end{aligned}$$

A.12 Proof of portfolio optimization, Proposition 1

The goal is to maximize

$$\begin{aligned} J(t,W_t,X_t,P_t,Q_t) = \sup_{\pi,c} E&\left[\varepsilon_1 \int_t^T e^{-\int_0^{s-t}\delta_{t+q}\mathrm{d}q} u(v(c_s,\tilde{\theta},P_s))\,\mathrm{d}s + \varepsilon_2 e^{-\int_0^{T-t}\delta_{t+q}\mathrm{d}q} u(v(W_T,\theta,P_T))\right] \\ \text{s.t. } \frac{\mathrm{d}W_t}{W_t} &= \frac{Q_t \mathbb{1}_{t\le T_R} - c_t}{W_t}\,\mathrm{d}t + (\pi^\intercal(\mu_A - r\mathbb{1}) + r)\,\mathrm{d}t + \pi^\intercal \Sigma_A\,\mathrm{d}Z_{A,t} \end{aligned}$$

for an agent that receives a wage flow in terms of stochastic endowment prices Q_t.

Rewriting the objective function as a recursive function for a small Δt

$$J(t,W_t,X_t,P_t,Q_t) = \sup_{\pi,c} E\begin{bmatrix} \varepsilon_1 e^{-\int_0^{\Delta t}\delta_{t+q}\mathrm{d}q} u(v(c_t,\tilde{\theta},P_t))\Delta t \\ + e^{-\int_0^{\Delta t}\delta_{t+q}\mathrm{d}q} J(t+\Delta t, W_{t+\Delta t}, X_{t+\Delta t}, P_{t+\Delta t}, Q_{t+\Delta t}) \end{bmatrix}$$

Then subtracting J from both sides, dividing by Δt and taking $\lim_{\Delta t\downarrow 0}$ yields

$$\begin{aligned} 0 =& \sup_{\pi,c} \varepsilon_1 u(v(c_t,\tilde{\theta},P_t)) - \delta J(t,W_t,X_t,P_t,Q_t) + \frac{\partial J}{\partial t} + J_W \frac{\partial W_t}{\partial t} + J_X^\intercal \frac{\partial X_t}{\partial t} + J_P^\intercal \frac{\partial P_t}{\partial t} \\ &+ \frac{1}{2} J_{WW} \left(\frac{\partial W_t}{\partial Z_{A,t}}\right)^\intercal \frac{\partial W_t}{\partial Z_{A,t}} + \frac{1}{2}\operatorname{tr}\left(J_{XX^\intercal} \frac{\partial X_t}{\partial Z_{X,t}} \left(\frac{\partial X_t}{\partial Z_{X,t}}\right)^\intercal\right) + \frac{1}{2}\operatorname{tr}\left(J_{PP^\intercal} \frac{\partial P_t}{\partial Z_{P,t}} \left(\frac{\partial P_t}{\partial Z_{P,t}}\right)^\intercal\right) \\ &+ J_{WX}^\intercal \frac{\partial X_t}{\partial Z_{X,t}} \rho_{XA} \frac{\partial W_t}{\partial Z_{A,t}} + J_{WP}^\intercal \frac{\partial P_t}{\partial Z_{P,t}} \rho_{PA} \frac{\partial W_t}{\partial Z_{A,t}} + \operatorname{tr}\left(J_{XP^\intercal}^\intercal \frac{\partial X_t}{\partial Z_{X,t}} \rho_{XP} \left(\frac{\partial P_t}{\partial Z_{P,t}}\right)^\intercal\right) \\ &+ J_Q \frac{\partial Q_t}{\partial t} + \frac{1}{2} J_{QQ} \left(\frac{\partial Q_t}{\partial Z_{Q,t}}\right)^\intercal \frac{\partial Q_t}{\partial Z_{Q,t}} \\ &+ J_{WQ} \left(\frac{\partial Q_t}{\partial Z_{Q,t}}\right)^\intercal \rho_{QA} \frac{\partial W_t}{\partial Z_{A,t}} + J_{XQ}^\intercal \frac{\partial X_t}{\partial Z_{X,t}} \rho_{XQ} \frac{\partial Q_t}{\partial Z_{Q,t}} + \operatorname{tr}\left(J_{QP}^\intercal \frac{\partial P_t}{\partial Z_{P,t}} \rho_{PQ} \frac{\partial Q_t}{\partial Z_{Q,t}}\right) \end{aligned}$$

Assuming a CRRA utility function and a Cobb-Douglas aggregator

$$
\begin{aligned}
0 = &\sup_{\pi,c} \varepsilon_1 \frac{\left(e^{(1-\tilde{\theta}^\intercal \mathbb{1})\log\left(1-\tilde{\theta}^\intercal \mathbb{1}\right)+\tilde{\theta}^\intercal \log\left(\tilde{\theta}\right)-\tilde{\theta}^\intercal \log(P_t)} c_t\right)^{1-\gamma}}{1-\gamma} - J_W c_t + J_W W_t \pi^\intercal (\mu_A - r\mathbb{1}) \\
&+ \frac{1}{2} J_{WW} W_t^2 \pi^\intercal \Sigma_A \Sigma_A^\intercal \pi + J_{WX}^\intercal \Sigma_X \rho_{XA} \Sigma_A^\intercal \pi W_t + J_{WP}^\intercal \operatorname{diag}(P_t) \Sigma_P \rho_{PA} \Sigma_A^\intercal \pi W_t \\
&- \delta J(t, W_t, X_t, P_t, Q_t) + \frac{\partial J}{\partial t} + J_W \left(W_t r + Q_t \mathbb{1}_{t \le T_R}\right) + J_X^\intercal \mu_X + \frac{1}{2} \operatorname{tr}\left(J_{XX^\intercal} \Sigma_X \Sigma_X^\intercal\right) \\
&+ J_P^\intercal \operatorname{diag}(P_t) \mu_P + \frac{1}{2} \operatorname{tr}\left(J_{PP^\intercal} \operatorname{diag}(P_t) \Sigma_P \Sigma_P^\intercal \operatorname{diag}(P_t)\right) + \operatorname{tr}\left(J_{XP^\intercal}^\intercal \Sigma_X \rho_{XP} \Sigma_P^\intercal \operatorname{diag}(P_t)\right) \\
&+ J_Q Q_t \mu_Q + \frac{1}{2} Q_t^2 J_{QQ} \Sigma_Q \Sigma_Q^\intercal + J_{WQ} Q_t \Sigma_Q \rho_{QA} \Sigma_A^\intercal \pi W_t \\
&+ Q_t J_{XQ}^\intercal \Sigma_X \rho_{XQ} \Sigma_Q^\intercal + J_{QP}^\intercal \operatorname{diag}(P_t) \Sigma_P \rho_{PQ} \Sigma_Q^\intercal Q_t
\end{aligned} \tag{A.27}
$$

The consumption problem below is concave due to the power utility term

$$
\sup_c \varepsilon_1 \frac{\left(e^{(1-\tilde{\theta}^\intercal \mathbb{1})\log\left(1-\tilde{\theta}^\intercal \mathbb{1}\right)+\tilde{\theta}^\intercal \log\left(\tilde{\theta}\right)-\tilde{\theta}^\intercal \log(P_t)} c_t\right)^{1-\gamma}}{1-\gamma} - J_W c_t
$$

with solution

$$
c_t^\star = \varepsilon_1^{\frac{1}{\gamma}} \left(e^{(1-\tilde{\theta}^\intercal \mathbb{1})\log\left(1-\tilde{\theta}^\intercal \mathbb{1}\right)+\tilde{\theta}^\intercal \log\left(\tilde{\theta}\right)-\tilde{\theta}^\intercal \log(P_t)}\right)^{\frac{1}{\gamma}-1} J_W^{-\frac{1}{\gamma}}.
$$

The investment problem is

$$
\begin{aligned}
\sup_\pi\ & J_W W_t \pi^\intercal (\mu_A - r\mathbb{1}) + \frac{1}{2} J_{WW} W_t^2 \pi^\intercal \Sigma_A \Sigma_A^\intercal \pi + J_{WX}^\intercal \Sigma_X \rho_{XA} \Sigma_A^\intercal \pi W_t \\
&+ J_{WP}^\intercal \operatorname{diag}(P_t) \Sigma_P \rho_{PA} \Sigma_A^\intercal \pi W_t + J_{WQ} Q_t \Sigma_Q \rho_{QA} \Sigma_A^\intercal \pi W_t
\end{aligned}
$$

with solution

$$
\pi_t^\star = \left(\Sigma_A \Sigma_A^\intercal\right)^{-1} \frac{\begin{pmatrix} J_W(\mu_A - r\mathbb{1}) + \Sigma_A \rho_{XA}^\intercal \Sigma_X^\intercal J_{WX} + \Sigma_A \rho_{PA}^\intercal \Sigma_P^\intercal \operatorname{diag}(P_t) J_{WP} \\ + \Sigma_A \rho_{QA}^\intercal \Sigma_Q^\intercal Q_t J_{WQ} \end{pmatrix}}{-J_{WW} W_t}.
$$

Second order conditions are satisfied given that $\Sigma_A \Sigma_A^\intercal$ is positive definite and J_{WW} is assumed to be negative, which ultimately follows from diminishing marginal utility of wealth in the utility function.

Substituting the solutions in the HJB yields

$$
\begin{aligned}
0 = &\frac{\gamma}{1-\gamma} \varepsilon_1^{\frac{1}{\gamma}} \left(e^{(1-\tilde{\theta}^\intercal \mathbb{1})\log\left(1-\tilde{\theta}^\intercal \mathbb{1}\right)+\tilde{\theta}^\intercal \log\left(\tilde{\theta}\right)-\tilde{\theta}^\intercal \log(P_t)}\right)^{\frac{1}{\gamma}-1} J_W^{1-\frac{1}{\gamma}} \\
&+ \frac{1}{2} \frac{\begin{pmatrix} \left(J_W(\mu_A - r\mathbb{1}) + \Sigma_A \rho_{XA}^\intercal \Sigma_X^\intercal J_{WX} + \Sigma_A \rho_{PA}^\intercal \Sigma_P^\intercal \operatorname{diag}(P_t) J_{WP} + \Sigma_A \rho_{QA}^\intercal \Sigma_Q^\intercal Q_t J_{WQ}\right)^\intercal \\ \cdot \left(\Sigma_A \Sigma_A^\intercal\right)^{-1} \\ \left(J_W(\mu_A - r\mathbb{1}) + \Sigma_A \rho_{XA}^\intercal \Sigma_X^\intercal J_{WX} + \Sigma_A \rho_{PA}^\intercal \Sigma_P^\intercal \operatorname{diag}(P_t) J_{WP} + \Sigma_A \rho_{QA}^\intercal \Sigma_Q^\intercal Q_t J_{WQ}\right) \end{pmatrix}}{-J_{WW}} \\
&- \delta J(t, W_t, X_t, P_t, Q_t) + \frac{\partial J}{\partial t} + J_W \left(W_t r + Q_t \mathbb{1}_{t \le T_R}\right) + J_X^\intercal \mu_X + \frac{1}{2} \operatorname{tr}\left(J_{XX^\intercal} \Sigma_X \Sigma_X^\intercal\right) \\
&+ J_P^\intercal \operatorname{diag}(P_t) \mu_P + \frac{1}{2} \operatorname{tr}\left(J_{PP^\intercal} \operatorname{diag}(P_t) \Sigma_P \Sigma_P^\intercal \operatorname{diag}(P_t)\right) + \operatorname{tr}\left(J_{XP^\intercal}^\intercal \Sigma_X \rho_{XP} \Sigma_P^\intercal \operatorname{diag}(P_t)\right)
\end{aligned}
$$

$$+ J_Q Q_t \mu_Q + \frac{1}{2} J_{QQ} Q_t^2 \Sigma_Q \Sigma_Q^\intercal + J_{XQ}^\intercal \Sigma_X \rho_{XQ} \Sigma_Q^\intercal Q_t + J_{QP}^\intercal \operatorname{diag}(P_t) \Sigma_P \rho_{PQ} \Sigma_Q^\intercal Q_t \quad \text{(A.28)}$$

with boundary condition $J(T, W_t, X_t, P_t, Q_t) = \varepsilon_2 \left(e^{(1-\theta^\intercal \mathbb{1}) \log(1-\theta^\intercal \mathbb{1}) + \theta^\intercal \log(\theta) - \theta^\intercal \log(P_t)}\right)^{1-\gamma} \frac{W_T^{1-\gamma}}{1-\gamma}$.

A.13 Proof of portfolio optimization with incomplete markets, Proposition 2

Continuing from Section A.12 and assuming that $Q_t = 0$, $\varepsilon_1 = 0$ and $\varepsilon_2 = 1$, consider the ansatz

$$J(t, W_t, X_t, P_t) = \frac{W_t^{1-\gamma}}{1-\gamma} f(t, X_t, P_t)^\gamma$$

Then the HJB becomes

$$\begin{aligned}
0 =& \frac{\partial f}{\partial t} \\
& - f(t, X_t, P_t) \frac{\delta - (1-\gamma)\left(r + \frac{1}{2} \frac{(\mu_A - r\mathbb{1})^\intercal \left(\Sigma_A \Sigma_A^\intercal\right)^{-1} (\mu_A - r\mathbb{1})}{\gamma}\right)}{\gamma} \\
& + \left(\frac{1-\gamma}{\gamma} (\mu_A - r\mathbb{1})^\intercal \left(\Sigma_A \Sigma_A^\intercal\right)^{-1} \Sigma_A \rho_{XA}^\intercal \Sigma_X^\intercal + \mu_X^\intercal\right) f_X \\
& + \left(\frac{1-\gamma}{\gamma} (\mu_A - r\mathbb{1})^\intercal \left(\Sigma_A \Sigma_A^\intercal\right)^{-1} \Sigma_A \rho_{PA}^\intercal \Sigma_P^\intercal \operatorname{diag}(P_t) + \mu_P^\intercal \operatorname{diag}(P_t)\right) f_P \\
& + \frac{1}{2}(1-\gamma) f(t, X_t, P_t)^{-1} f_X^\intercal \Sigma_X \left(\rho_{XA} \rho_{XA}^\intercal - I\right) \Sigma_X^\intercal f_X \\
& + \frac{1}{2}(1-\gamma) f(t, X_t, P_t)^{-1} f_P^\intercal \operatorname{diag}(P_t) \Sigma_P \left(\rho_{PA} \rho_{PA}^\intercal - I\right) \Sigma_P^\intercal \operatorname{diag}(P_t) f_P \\
& + (1-\gamma) f(t, X_t, P_t)^{-1} f_X^\intercal \Sigma_X \left(\rho_{XA} \rho_{PA}^\intercal - \rho_{XP}\right) \Sigma_P^\intercal \operatorname{diag}(P_t) f_P \\
& + \frac{1}{2} \operatorname{tr}\left(f_{XX^\intercal} \Sigma_X \Sigma_X^\intercal\right) \\
& + \frac{1}{2} \operatorname{tr}\left(f_{PP^\intercal} \operatorname{diag}(P_t) \Sigma_P \Sigma_P^\intercal \operatorname{diag}(P_t)\right) \\
& + \operatorname{tr}\left(f_{XP^\intercal}^\intercal \Sigma_X \rho_{XP} \Sigma_P^\intercal \operatorname{diag}(P_t)\right)
\end{aligned}$$

with boundary condition $f(T, X_t, P_t) = \left(e^{(1-\theta^\intercal \mathbb{1}) \log(1-\theta^\intercal \mathbb{1}) + \theta^\intercal \log(\theta) - \theta^\intercal \log(P_T)}\right)^{\frac{1}{\gamma}-1}$.

$$\begin{aligned}
\pi_t^\star =& \left(\Sigma_A \Sigma_A^\intercal\right)^{-1} \frac{\mu_A - r\mathbb{1}}{\gamma} \\
& + \left(\Sigma_A \Sigma_A^\intercal\right)^{-1} \Sigma_A \rho_{XA}^\intercal \Sigma_X^\intercal f_X f(t, X_t, P_t)^{-1} \\
& + \left(\Sigma_A \Sigma_A^\intercal\right)^{-1} \Sigma_A \rho_{PA}^\intercal \Sigma_P^\intercal \operatorname{diag}(P_t) f_P f(t, X_t, P_t)^{-1}
\end{aligned}$$

Further use the ansatz

$$f(t, X_t, P_t) = \left(e^{(1-\theta^\intercal \mathbb{1}) \log(1-\theta^\intercal \mathbb{1}) + \theta^\intercal \log(\theta) - \theta^\intercal \log(P_t)}\right)^{\frac{1}{\gamma}-1} h(t, X_t)$$

then

$$
\begin{aligned}
\frac{\partial h}{\partial t} =& h(t,X_t)\left(\frac{\delta}{\gamma}-\left(\frac{1}{\gamma}-1\right)\left(\begin{array}{l}
r+\frac{1}{2}\frac{(\mu_A-r\mathbb{1})^\intercal\left(\Sigma_A\Sigma_A^\intercal\right)^{-1}(\mu_A-r\mathbb{1})}{\gamma} \\
-\theta^\intercal\left(\mu_P+\left(\frac{1}{\gamma}-1\right)\Sigma_P\rho_{PA}\Sigma_A^\intercal\left(\Sigma_A\Sigma_A^\intercal\right)^{-1}(\mu_A-r\mathbb{1})\right) \\
+\frac{1}{2}\left(\frac{1}{\gamma}-1\right)\theta^\intercal\Sigma_P\left(\gamma I+(1-\gamma)\rho_{PA}\rho_{PA}^\intercal\right)\Sigma_P^\intercal\theta \\
+\frac{1}{2}\operatorname{tr}\left(\operatorname{diag}(\theta)\Sigma_P\Sigma_P^\intercal\right)
\end{array}\right)\right) \\
&-h_X^\intercal\left(\mu_X+\left(\frac{1}{\gamma}-1\right)\Sigma_X\left(\rho_{XA}\Sigma_A^\intercal\left(\Sigma_A\Sigma_A^\intercal\right)^{-1}(\mu_A-r\mathbb{1})-\left(\gamma\rho_{XP}+(1-\gamma)\rho_{XA}\rho_{PA}^\intercal\right)\Sigma_P^\intercal\theta\right)\right) \\
&-\frac{1}{2}\operatorname{tr}\left(h_{XX^\intercal}\Sigma_X\Sigma_X^\intercal\right) \\
&-\frac{1}{2}(1-\gamma)h_X^\intercal\Sigma_X\left(\rho_{XA}\rho_{XA}^\intercal-I\right)\Sigma_X^\intercal h_X\frac{1}{h(t,X_t)}
\end{aligned}
$$

with boundary condition $h(T,X_t)=1$.

The optimal investment strategy becomes

$$
\begin{aligned}
\pi_t^\star =& \left(\Sigma_A\Sigma_A^\intercal\right)^{-1}\frac{\mu_A-r\mathbb{1}}{\gamma} \\
&+\left(\Sigma_A\Sigma_A^\intercal\right)^{-1}\Sigma_A\rho_{XA}^\intercal\Sigma_X^\intercal\frac{h_X}{h(t,X_t)} \\
&-\left(\frac{1}{\gamma}-1\right)\left(\Sigma_A\Sigma_A^\intercal\right)^{-1}\Sigma_A\rho_{PA}^\intercal\Sigma_P^\intercal\theta
\end{aligned}
$$

The PDE for $h(t,X_t)$ is a particular case of the generic PDE from Lemma 3 parametrized as

$$
g\left(\begin{array}{l}
t,X_t; \\
R=-\frac{\delta}{\gamma}+\left(\frac{1}{\gamma}-1\right)\left(\begin{array}{l}
r+\frac{1}{2}\frac{(\mu_A-r\mathbb{1})^\intercal\left(\Sigma_A\Sigma_A^\intercal\right)^{-1}(\mu_A-r\mathbb{1})}{\gamma} \\
-\theta^\intercal\left(\mu_P+\left(\frac{1}{\gamma}-1\right)\Sigma_P\rho_{PA}\Sigma_A^\intercal\left(\Sigma_A\Sigma_A^\intercal\right)^{-1}(\mu_A-r\mathbb{1})\right) \\
+\frac{1}{2}\left(\frac{1}{\gamma}-1\right)\theta^\intercal\Sigma_P\left(\gamma I+(1-\gamma)\rho_{PA}\rho_{PA}^\intercal\right)\Sigma_P^\intercal\theta \\
+\frac{1}{2}\operatorname{tr}\left(\operatorname{diag}(\theta)\Sigma_P\Sigma_P^\intercal\right)
\end{array}\right), \\
B=\mu_X+\left(\frac{1}{\gamma}-1\right)\Sigma_X\left(\rho_{XA}\Sigma_A^\intercal\left(\Sigma_A\Sigma_A^\intercal\right)^{-1}(\mu_A-r\mathbb{1})-\left(\gamma\rho_{XP}+(1-\gamma)\rho_{XA}\rho_{PA}^\intercal\right)\Sigma_P^\intercal\theta\right), \\
C=\Sigma_X\left(\gamma I+(1-\gamma)\rho_{XA}\rho_{XA}^\intercal\right)\Sigma_X^\intercal, \\
D=\Sigma_X\Sigma_X^\intercal
\end{array}\right)
$$

with boundary condition $g(T,X_t)=1$.

The matrices C and D are at least positive semi-definite. In the case of C, notice that when formulated in this way

$$
C=\Sigma_X\left(\rho_{XA}\rho_{XA}^\intercal+\gamma\left(I-\rho_{XA}\rho_{XA}^\intercal\right)\right)\Sigma_X^\intercal
$$

we only need to show that $I - \rho_{XA}\rho_{XA}^\intercal$ is positive semi-definite. Since the Brownian driver for $Z_{X,t}$ and $Z_{A,t}$ are orthogonal standard normal vectors, covariance and correlation coincide and we have that

$$\rho_{XX} = I = \rho_{XA}\rho_{XA}^\intercal + \rho_{X\bar{A}}\rho_{X\bar{A}}^\intercal$$

where $Z_{\bar{A},t}$ is an orthogonal standard normal vector complementary to $Z_{A,t}$. Thus

$$I - \rho_{XA}\rho_{XA}^\intercal = \rho_{X\bar{A}}\rho_{X\bar{A}}^\intercal$$

is positive semi-definite.

Section A.5 explains how to reduce the generic PDE to a system of Riccati ODEs by parametrizing A, B, C, D quadratically, which in this case can be constructed from the following building blocks

$$\begin{aligned}
\delta &= {}_{\delta}\alpha + {}_{\delta}\beta_p\, X^p + X_p\, \eta_h{}^p\, {}_{\delta}\omega^h{}_m\, \eta^m{}_q\, X^q \\
r &= {}_{r}\alpha + {}_{r}\beta_p\, X^p + X_p\, \eta_h{}^p\, {}_{r}\omega^h{}_m\, \eta^m{}_q\, X^q \\
(\mu_A - r\mathbb{1})^\intercal \left(\Sigma_A\Sigma_A^\intercal\right)^{-1}(\mu_A - r\mathbb{1}) &= {}_{\Lambda}\tilde{\alpha} + {}_{\Lambda}\tilde{\beta}_p\, X^p + X_p\, \eta_h{}^p\, {}_{\Lambda}\tilde{\omega}^h{}_m\, \eta^m{}_q\, X^q \\
(\mu_P)^k &= {}_{P}\alpha^k + {}_{P}\beta^k{}_p\, X^p + X_p\, \eta_h{}^p\, {}_{P}\omega^{kh}{}_m\, \eta^m{}_q\, X^q \\
\left(\Sigma_P\rho_{PA}\Sigma_A^\intercal\left(\Sigma_A\Sigma_A^\intercal\right)^{-1}(\mu_A - r\mathbb{1})\right)^k &= {}_{\Sigma_P A}\alpha^k + {}_{\Sigma_P A}\beta^k{}_p\, X^p + X_p\, \eta_h{}^p\, {}_{\Sigma_P A}\omega^{kh}{}_m\, \eta^m{}_q\, X^q \\
\left(\Sigma_P\left(\gamma I + (1-\gamma)\rho_{PA}\rho_{PA}^\intercal\right)\Sigma_P^\intercal\right)^k{}_l &= {}_{\Sigma_P}\tilde{\alpha}^k{}_l + {}_{\Sigma_P}\tilde{\beta}^k{}_{lp}\, X^p + X_p\, \eta_h{}^p\, {}_{\Sigma_P}\tilde{\omega}^k{}_l{}^h{}_m\, \eta^m{}_q\, X^q \\
\left(\Sigma_P\Sigma_P^\intercal\right)^k{}_l &= {}_{\Sigma_P}\alpha^k{}_l + {}_{\Sigma_P}\beta^k{}_{lp}\, X^p + X_p\, \eta_h{}^p\, {}_{\Sigma_P}\omega^k{}_l{}^h{}_m\, \eta^m{}_q\, X^q \\
(\mu_X)^k &= {}_{X}\alpha^k - {}_{X}\beta^k{}_p\, X^p + X_p\, \eta_h{}^p\, {}_{X}\omega^{kh}{}_m\, \eta^m{}_q\, X^q \\
\left(\Sigma_X\rho_{XA}\Sigma_A^\intercal\left(\Sigma_A\Sigma_A^\intercal\right)^{-1}(\mu_A - r\mathbb{1})\right)^k &= {}_{\Sigma_X A}\alpha^k + {}_{\Sigma_X A}\beta^k{}_p\, X^p + X_p\, \eta_h{}^p\, {}_{\Sigma_X A}\omega^{kh}{}_m\, \eta^m{}_q\, X^q \\
\left(\Sigma_X\left(\gamma\rho_{XP} + (1-\gamma)\rho_{XA}\rho_{PA}^\intercal\right)\Sigma_P^\intercal\right)^k{}_l &= {}_{\Sigma_X\Sigma_P}\alpha^k{}_l + {}_{\Sigma_X\Sigma_P}\beta^k{}_{lp}\, X^p + X_p\, \eta_h{}^p\, {}_{\Sigma_X\Sigma_P}\omega^k{}_l{}^h{}_m\, \eta^m{}_q\, X^q \\
\left(\Sigma_X\left(\gamma I + (1-\gamma)\rho_{XA}\rho_{XA}^\intercal\right)\Sigma_X^\intercal\right)^k{}_l &= {}_{\Sigma_X}\tilde{\alpha}^k{}_l + {}_{\Sigma_X}\tilde{\beta}^k{}_{lp}\, X^p + X_p\, \eta_h{}^p\, {}_{\Sigma_X}\tilde{\omega}^k{}_l{}^h{}_m\, \eta^m{}_q\, X^q \\
\left(\Sigma_X\Sigma_X^\intercal\right)^k{}_l &= {}_{\Sigma_X}\alpha^k{}_l + {}_{\Sigma_X}\beta^k{}_{lp}\, X^p + X_p\, \eta_h{}^p\, {}_{\Sigma_X}\omega^k{}_l{}^h{}_m\, \eta^m{}_q\, X^q
\end{aligned}$$

Section A.6 shows how to explicitly solve the diagonalized version of the aforementioned Riccati ODEs for an ample range of cases.

A.14 Proof of portfolio optimization with complete markets, Proposition 3

Continuing from Section A.12, considering the ansatz

$$J(t, W_t, X_t, P_t, Q_t) = \frac{(W_t + \Upsilon(t, X_t, Q_t))^{1-\gamma}}{1-\gamma} f(t, X_t, P_t)^\gamma$$

then the HJB becomes

$$
\begin{aligned}
0 =& \frac{J(t,W_t,X_t,P_t,Q_t)}{f(t,X_t,P_t)}\gamma \left(\begin{aligned}
&\frac{\partial f}{\partial t} + \varepsilon_1^{\frac{1}{\gamma}} \left(e^{(1-\tilde{\theta}^\intercal \mathbb{1})\log\left(1-\tilde{\theta}^\intercal \mathbb{1}\right)+\tilde{\theta}^\intercal \log\left(\tilde{\theta}\right)-\tilde{\theta}^\intercal \log(P_t)}\right)^{\frac{1}{\gamma}-1}\\
&- f(t,X_t,P_t)\left(\frac{\delta}{\gamma} - \frac{1-\gamma}{\gamma}\left(r + \frac{1}{2}\frac{(\mu_A - r\mathbb{1})^\intercal (\Sigma_A\Sigma_A^\intercal)^{-1}(\mu_A - r\mathbb{1})}{\gamma}\right)\right)\\
&+ \left(\mu_X^\intercal + \frac{1-\gamma}{\gamma}(\mu_A - r\mathbb{1})^\intercal (\Sigma_A\Sigma_A^\intercal)^{-1}\Sigma_A\rho_{XA}^\intercal\Sigma_X^\intercal\right) f_X\\
&+ \left(\mu_P^\intercal + \frac{1-\gamma}{\gamma}(\mu_A - r\mathbb{1})^\intercal (\Sigma_A\Sigma_A^\intercal)^{-1}\Sigma_A\rho_{PA}^\intercal\Sigma_P^\intercal\right) \operatorname{diag}(P_t) f_P\\
&+ (1-\gamma)\frac{1}{2}\frac{f_X^\intercal \Sigma_X\left(\rho_{XA}\Sigma_A^\intercal (\Sigma_A\Sigma_A^\intercal)^{-1}\Sigma_A\rho_{XA}^\intercal - I\right)\Sigma_X^\intercal f_X}{f(t,X_t,P_t)}\\
&+ (1-\gamma)\frac{1}{2}\frac{f_P^\intercal \operatorname{diag}(P_t)\Sigma_P\left(\rho_{PA}\Sigma_A^\intercal (\Sigma_A\Sigma_A^\intercal)^{-1}\Sigma_A\rho_{PA}^\intercal - I\right)\Sigma_P^\intercal \operatorname{diag}(P_t) f_P}{f(t,X_t,P_t)}\\
&+ (1-\gamma)\frac{f_X^\intercal \Sigma_X\left(\rho_{XA}\Sigma_A^\intercal (\Sigma_A\Sigma_A^\intercal)^{-1}\Sigma_A\rho_{PA}^\intercal - \rho_{XP}\right)\Sigma_P^\intercal \operatorname{diag}(P_t) f_P}{f(t,X_t,P_t)}\\
&+ \frac{1}{2}\operatorname{tr}\left(f_{XX^\intercal}\Sigma_X\Sigma_X^\intercal\right) + \frac{1}{2}\operatorname{tr}\left(f_{PP^\intercal}\operatorname{diag}(P_t)\Sigma_P\Sigma_P^\intercal\operatorname{diag}(P_t)\right)\\
&+ \operatorname{tr}\left(f_{XP^\intercal}^\intercal \Sigma_X\rho_{XP}\Sigma_P^\intercal\operatorname{diag}(P_t)\right)
\end{aligned}\right)\\
&+ J_W \left(\begin{aligned}
&\frac{\partial \Upsilon}{\partial t} - \Upsilon(t,X_t,Q_t)r + Q_t\mathbb{1}_{t\le T_R}\\
&+ \left(\mu_X^\intercal - (\mu_A - r\mathbb{1})^\intercal (\Sigma_A\Sigma_A^\intercal)^{-1}\Sigma_A\rho_{XA}^\intercal\Sigma_X^\intercal\right)\Upsilon_X\\
&+ \gamma f(t,X_t,P_t)^{-1} f_X^\intercal \Sigma_X\left(I - \rho_{XA}\Sigma_A^\intercal (\Sigma_A\Sigma_A^\intercal)^{-1}\Sigma_A\rho_{XA}^\intercal\right)\Sigma_X^\intercal\Upsilon_X\\
&+ \gamma f(t,X_t,P_t)^{-1} f_P^\intercal \operatorname{diag}(P_t)\Sigma_P\left(\rho_{XP}^\intercal - \rho_{PA}\Sigma_A^\intercal (\Sigma_A\Sigma_A^\intercal)^{-1}\Sigma_A\rho_{XA}^\intercal\right)\Sigma_X^\intercal\Upsilon_X\\
&+ \frac{1}{2}\operatorname{tr}\left(\Upsilon_{XX^\intercal}\Sigma_X\Sigma_X^\intercal\right)\\
&+ \Upsilon_{XQ}^\intercal\Sigma_X\rho_{XQ}\Sigma_Q^\intercal Q_t\\
&+ \left(\mu_Q - (\mu_A - r\mathbb{1})^\intercal (\Sigma_A\Sigma_A^\intercal)^{-1}\Sigma_A\rho_{QA}^\intercal\Sigma_Q^\intercal\right)Q_t\Upsilon_Q\\
&+ \gamma f(t,X_t,P_t)^{-1} f_X^\intercal \Sigma_X\left(\rho_{XQ} - \rho_{XA}\Sigma_A^\intercal (\Sigma_A\Sigma_A^\intercal)^{-1}\Sigma_A\rho_{QA}^\intercal\right)\Sigma_Q^\intercal Q_t\Upsilon_Q\\
&+ \gamma f(t,X_t,P_t)^{-1} f_P^\intercal \operatorname{diag}(P_t)\Sigma_P\left(\rho_{PQ} - \rho_{PA}\Sigma_A^\intercal (\Sigma_A\Sigma_A^\intercal)^{-1}\Sigma_A\rho_{QA}^\intercal\right)\Sigma_Q^\intercal Q_t\Upsilon_Q\\
&+ \frac{1}{2}\Upsilon_{QQ}Q_t\Sigma_Q\Sigma_Q^\intercal Q_t\\
&+ \frac{1}{W_t + \Upsilon(t,X_t,Q_t)}\gamma\frac{1}{2}\Upsilon_X^\intercal\Sigma_X\left(\rho_{XA}\Sigma_A^\intercal (\Sigma_A\Sigma_A^\intercal)^{-1}\Sigma_A\rho_{XA}^\intercal - I\right)\Sigma_X^\intercal\Upsilon_X\\
&+ \frac{1}{W_t + \Upsilon(t,X_t,Q_t)}\gamma\Upsilon_X^\intercal\Sigma_X\left(\rho_{XA}\Sigma_A^\intercal (\Sigma_A\Sigma_A^\intercal)^{-1}\Sigma_A\rho_{QA}^\intercal - \rho_{XQ}\right)\Sigma_Q^\intercal Q_t\Upsilon_Q\\
&+ \frac{1}{W_t + \Upsilon(t,X_t,Q_t)}\gamma\frac{1}{2}\Upsilon_Q Q_t\Sigma_Q\left(\rho_{QA}\Sigma_A^\intercal (\Sigma_A\Sigma_A^\intercal)^{-1}\Sigma_A\rho_{QA}^\intercal - I\right)\Sigma_Q^\intercal Q_t\Upsilon_Q
\end{aligned}\right)
\end{aligned} \tag{A.29}
$$

To remove the dependency on W_t, the endowment payoffs Q_t as well as state indicators X_t

must be tradeable (Assumption 2)

$$
\begin{aligned}
0 =& \Upsilon_X^\intercal \Sigma_X \left(\rho_{XA}\Sigma_A^\intercal \left(\Sigma_A\Sigma_A^\intercal\right)^{-1}\Sigma_A\rho_{XA}^\intercal - I\right)\Sigma_X^\intercal \Upsilon_X \\
0 =& \Upsilon_X^\intercal \Sigma_X \left(\rho_{XA}\Sigma_A^\intercal \left(\Sigma_A\Sigma_A^\intercal\right)^{-1}\Sigma_A\rho_{QA}^\intercal - \rho_{XQ}\right)\Sigma_Q^\intercal Q_t \Upsilon_Q \\
0 =& \Upsilon_Q Q_t \Sigma_Q \left(\rho_{QA}\Sigma_A^\intercal \left(\Sigma_A\Sigma_A^\intercal\right)^{-1}\Sigma_A\rho_{QA}^\intercal - I\right)\Sigma_Q^\intercal Q_t \Upsilon_Q.
\end{aligned}
$$

The terms multiplying $J(t, W_t, X_t, P_t, Q_t)$ in the PDE above must add up to zero as well as the terms multiplying J_W. These condition gives rise to PDEs for $f(t, X_t, P_t)$ and $\Upsilon(t, X_t, Q_t)$ respectively. The associated optimal controls are

$$
c_t^\star = \varepsilon_1^{\frac{1}{\gamma}}\left(e^{(1-\tilde\theta^\intercal \mathbb{1})\log\left(1-\tilde\theta^\intercal\mathbb{1}\right)+\tilde\theta^\intercal\log\left(\tilde\theta\right)-\tilde\theta^\intercal\log(P_t)}\right)^{\frac{1}{\gamma}-1} f(t,X_t,P_t)^{-1}\left(W_t+\Upsilon(t,X_t,Q_t)\right) \tag{A.30}
$$

$$
\begin{aligned}
\pi_t^\star =& \left(\Sigma_A\Sigma_A^\intercal\right)^{-1}\frac{(\mu_A - r\mathbb{1})}{\gamma}\frac{W_t+\Upsilon(t,X_t,Q_t)}{W_t} \\
&+ \left(\Sigma_A\Sigma_A^\intercal\right)^{-1}\Sigma_A\rho_{XA}^\intercal\Sigma_X^\intercal\left(f(t,X_t,P_t)^{-1}f_X\frac{W_t+\Upsilon(t,X_t,Q_t)}{W_t} - \frac{\Upsilon_X}{W_t}\right) \\
&+ \left(\Sigma_A\Sigma_A^\intercal\right)^{-1}\Sigma_A\rho_{PA}^\intercal\Sigma_P^\intercal \operatorname{diag}(P_t) f(t,X_t,P_t)^{-1}f_P\frac{W_t+\Upsilon(t,X_t,Q_t)}{W_t} \\
&- \left(\Sigma_A\Sigma_A^\intercal\right)^{-1}\Sigma_A\rho_{QA}^\intercal\Sigma_Q^\intercal Q_t\Upsilon_Q\frac{1}{W_t}
\end{aligned} \tag{A.31}
$$

Solution to $f(t, X_t, P_t)$. The terms multiplying $J(t, W_t, X_t, P_t, Q_t)$ in (A.29) give rise to a PDE for $f(t, X_t, P_t)$

$$
0 = \varepsilon_1^{\frac{1}{\gamma}}\left(e^{(1-\tilde\theta^\intercal \mathbb{1})\log\left(1-\tilde\theta^\intercal\mathbb{1}\right)+\tilde\theta^\intercal\log\left(\tilde\theta\right)-\tilde\theta^\intercal\log(P_t)}\right)^{\frac{1}{\gamma}-1} + \frac{\partial f}{\partial t} \tag{A.32}
$$

$$
- f(t,X_t,P_t)\frac{\delta - (1-\gamma)\left(r + \frac{1}{2}\frac{(\mu_A - r\mathbb{1})^\intercal\left(\Sigma_A\Sigma_A^\intercal\right)^{-1}(\mu_A - r\mathbb{1})}{\gamma}\right)}{\gamma} \tag{A.33}
$$

$$
+ \left(\mu_X^\intercal + \frac{1-\gamma}{\gamma}(\mu_A - r\mathbb{1})^\intercal\left(\Sigma_A\Sigma_A^\intercal\right)^{-1}\Sigma_A\rho_{XA}^\intercal\Sigma_X^\intercal\right) f_X \tag{A.34}
$$

$$
+ \left(\mu_P^\intercal + \frac{1-\gamma}{\gamma}(\mu_A - r\mathbb{1})^\intercal\left(\Sigma_A\Sigma_A^\intercal\right)^{-1}\Sigma_A\rho_{PA}^\intercal\Sigma_P^\intercal\right) \operatorname{diag}(P_t) f_P \tag{A.35}
$$

$$
+ \frac{1}{2}(1-\gamma) f(t,X_t,P_t)^{-1} f_X^\intercal \Sigma_X\left(\rho_{XA}\Sigma_A^\intercal\left(\Sigma_A\Sigma_A^\intercal\right)^{-1}\Sigma_A\rho_{XA}^\intercal - I\right)\Sigma_X^\intercal f_X \tag{A.36}
$$

$$
+ \frac{1}{2}(1-\gamma) f(t,X_t,P_t)^{-1} f_P^\intercal \operatorname{diag}(P_t)\Sigma_P\left(\rho_{PA}\Sigma_A^\intercal\left(\Sigma_A\Sigma_A^\intercal\right)^{-1}\Sigma_A\rho_{PA}^\intercal - I\right)\Sigma_P^\intercal \operatorname{diag}(P_t) f_P \tag{A.37}
$$

$$
+ (1-\gamma) f(t,X_t,P_t)^{-1} f_X^\intercal \Sigma_X\left(\rho_{XA}\Sigma_A^\intercal\left(\Sigma_A\Sigma_A^\intercal\right)^{-1}\Sigma_A\rho_{PA}^\intercal - \rho_{XP}\right)\Sigma_P^\intercal \operatorname{diag}(P_t) f_P \tag{A.38}
$$

$$
+ \frac{1}{2}\operatorname{tr}\left(f_{XX^\intercal}\Sigma_X\Sigma_X^\intercal\right) \tag{A.39}
$$

$$
+ \frac{1}{2}\operatorname{tr}\left(f_{PP^\intercal}\operatorname{diag}(P_t)\Sigma_P\Sigma_P^\intercal\operatorname{diag}(P_t)\right) \tag{A.40}
$$

$$
+ \operatorname{tr}\left(f_{XP^\intercal}^\intercal\Sigma_X\rho_{XP}\Sigma_P^\intercal\operatorname{diag}(P_t)\right) \tag{A.41}
$$

with boundary condition $f(T, X_t, P_t) = \varepsilon_2^{\frac{1}{\gamma}} \left(e^{(1-\theta^\intercal \mathbb{1})\log(1-\theta^\intercal \mathbb{1})+\theta^\intercal \log(\theta)-\theta^\intercal \log(P_T)}\right)^{\frac{1}{\gamma}-1}$.

Assuming that state indicators X_t and consumption prices P_t are tradeable (Assumption 2), we have

$$
\begin{aligned}
0 =& \Sigma_X \left(\rho_{XA}\Sigma_A^\intercal \left(\Sigma_A\Sigma_A^\intercal\right)^{-1} \Sigma_A\rho_{XA}^\intercal - I\right)\Sigma_X^\intercal \\
0 =& \Sigma_P \left(\rho_{PA}\Sigma_A^\intercal \left(\Sigma_A\Sigma_A^\intercal\right)^{-1} \Sigma_A\rho_{PA}^\intercal - I\right)\Sigma_P^\intercal \\
0 =& \Sigma_X \left(\rho_{XA}\Sigma_A^\intercal \left(\Sigma_A\Sigma_A^\intercal\right)^{-1} \Sigma_A\rho_{PA}^\intercal - \rho_{XP}\right)\Sigma_P^\intercal
\end{aligned}
$$

and considering the ansatz

$$
\begin{aligned}
f(t, X_t, P_t) =& \varepsilon_2^{\frac{1}{\gamma}} \left(e^{(1-\theta^\intercal \mathbb{1})\log(1-\theta^\intercal \mathbb{1})+\theta^\intercal \log(\theta)-\theta^\intercal \log(P_t)}\right)^{\frac{1}{\gamma}-1} h(t, X_t; T, \theta) \\
&+ \varepsilon_1^{\frac{1}{\gamma}} \left(e^{(1-\tilde{\theta}^\intercal \mathbb{1})\log\left(1-\tilde{\theta}^\intercal \mathbb{1}\right)+\tilde{\theta}^\intercal \log\left(\tilde{\theta}\right)-\tilde{\theta}^\intercal \log(P_t)}\right)^{\frac{1}{\gamma}-1} \int_t^T \tilde{h}(t, X_t; s)\,\mathrm{d}s
\end{aligned}
$$

we can replace them in the PDE above to arrive at

$$
\begin{aligned}
0 =& \varepsilon_1^{\frac{1}{\gamma}} \left(e^{(1-\tilde{\theta}^\intercal \mathbb{1})\log\left(1-\tilde{\theta}^\intercal \mathbb{1}\right)+\tilde{\theta}^\intercal \log\left(\tilde{\theta}\right)-\tilde{\theta}^\intercal \log(P_t)}\right)^{\frac{1}{\gamma}-1} \\
&\cdot \left(
\begin{aligned}
&\underbrace{\tilde{h}(t, X_t; t)}_{1} - \tilde{h}(t, X_t; t) \\
&- \int_t^T \underbrace{\left(
\begin{aligned}
&- \frac{\partial \tilde{h}}{\partial t} \\
&+ \tilde{h}(t, X_t; s)\frac{\delta - (1-\gamma)\left(r + \frac{1}{2}\frac{(\mu_A - r\mathbb{1})^\intercal \left(\Sigma_A\Sigma_A^\intercal\right)^{-1}(\mu_A - r\mathbb{1})}{\gamma}\right)}{\gamma} \\
&+ \tilde{h}(t, X_t; s)\left(\mu_P^\intercal + \frac{1-\gamma}{\gamma}(\mu_A - r\mathbb{1})^\intercal \left(\Sigma_A\Sigma_A^\intercal\right)^{-1} \Sigma_A\rho_{PA}^\intercal\Sigma_P^\intercal\right)\tilde{\theta}\left(\frac{1}{\gamma} - 1\right) \\
&- \tilde{h}(t, X_t; s)\frac{1}{2}\left(\frac{1}{\gamma} - 1\right)^2 \tilde{\theta}^\intercal \Sigma_P\Sigma_P^\intercal\tilde{\theta} \\
&- \tilde{h}(t, X_t; s)\frac{1}{2}\left(\frac{1}{\gamma} - 1\right)\mathrm{tr}\left(\mathrm{diag}\left(\tilde{\theta}\right)\Sigma_P\Sigma_P^\intercal\right) \\
&- \left(\mu_X^\intercal + \frac{1-\gamma}{\gamma}(\mu_A - r\mathbb{1})^\intercal \left(\Sigma_A\Sigma_A^\intercal\right)^{-1} \Sigma_A\rho_{XA}^\intercal\Sigma_X^\intercal\right)\tilde{h}_X \\
&+ \left(\frac{1}{\gamma} - 1\right)\tilde{h}_X^\intercal \Sigma_X\rho_{XP}\Sigma_P^\intercal\tilde{\theta} \\
&- \frac{1}{2}\mathrm{tr}\left(\tilde{h}_{XX^\intercal}\Sigma_X\Sigma_X^\intercal\right)
\end{aligned}
\right)}_{0} \mathrm{d}s
\end{aligned}
\right) \\
&- \varepsilon_2^{\frac{1}{\gamma}} \left(e^{(1-\theta^\intercal \mathbb{1})\log(1-\theta^\intercal \mathbb{1})+\theta^\intercal \log(\theta)-\theta^\intercal \log(P_t)}\right)^{\frac{1}{\gamma}-1}
\end{aligned}
$$

$$
\cdot \left(\begin{array}{l}
-\dfrac{\partial h}{\partial t} \\
+h(t,X_t;T,\theta)\dfrac{\delta-(1-\gamma)\left(r+\frac{1}{2}\frac{(\mu_A-r\mathbb{1})^\intercal\left(\Sigma_A\Sigma_A^\intercal\right)^{-1}(\mu_A-r\mathbb{1})}{\gamma}\right)}{\gamma} \\
+h(t,X_t;T,\theta)\left(\mu_P^\intercal+\dfrac{1-\gamma}{\gamma}(\mu_A-r\mathbb{1})^\intercal\left(\Sigma_A\Sigma_A^\intercal\right)^{-1}\Sigma_A\rho_{PA}^\intercal\Sigma_P^\intercal\right)\theta\left(\dfrac{1}{\gamma}-1\right) \\
-h(t,X_t;T,\theta)\dfrac{1}{2}\left(\dfrac{1}{\gamma}-1\right)^2\theta^\intercal\Sigma_P\Sigma_P^\intercal\theta \\
-h(t,X_t;T,\theta)\dfrac{1}{2}\left(\dfrac{1}{\gamma}-1\right)\mathrm{tr}\left(\mathrm{diag}\left(\theta\right)\Sigma_P\Sigma_P^\intercal\right) \\
-\left(\mu_X^\intercal+\dfrac{1-\gamma}{\gamma}(\mu_A-r\mathbb{1})^\intercal\left(\Sigma_A\Sigma_A^\intercal\right)^{-1}\Sigma_A\rho_{XA}^\intercal\Sigma_X^\intercal\right)h_X \\
+\left(\dfrac{1}{\gamma}-1\right)h_X^\intercal\Sigma_X\rho_{XP}\Sigma_P^\intercal\theta \\
-\dfrac{1}{2}\mathrm{tr}\left(h_{XX^\intercal}\Sigma_X\Sigma_X^\intercal\right)
\end{array} \right)
$$

We can see that, on the one hand the terms multiplying $\left(e^{(1-\tilde{\theta}^\intercal\mathbb{1})\log\left(1-\tilde{\theta}^\intercal\mathbb{1}\right)+\tilde{\theta}^\intercal\log\left(\tilde{\theta}\right)-\tilde{\theta}^\intercal\log(P_t)}\right)^{\frac{1}{\gamma}-1}$ must be zero, and on the other hand the terms multiplying $\left(e^{(1-\theta^\intercal\mathbb{1})\log(1-\theta^\intercal\mathbb{1})+\theta^\intercal\log(\theta)-\theta^\intercal\log(P_t)}\right)^{\frac{1}{\gamma}-1}$ must be zero, for the PDE to hold for any P_t, $\tilde{\theta}$ and θ. Thus, we can reformulate the former PDE into this one for $\tilde{h}$

$$
\begin{aligned}
\frac{\partial\tilde{h}}{\partial t}=&\tilde{h}(t,X_t;T)\left(\begin{array}{l}
\dfrac{\delta-(1-\gamma)\left(r+\frac{1}{2}\frac{(\mu_A-r\mathbb{1})^\intercal\left(\Sigma_A\Sigma_A^\intercal\right)^{-1}(\mu_A-r\mathbb{1})}{\gamma}\right)}{\gamma} \\
+\left(\dfrac{1}{\gamma}-1\right)\left(\begin{array}{l}
\left(\mu_P^\intercal+\dfrac{1-\gamma}{\gamma}(\mu_A-r\mathbb{1})^\intercal\left(\Sigma_A\Sigma_A^\intercal\right)^{-1}\Sigma_A\rho_{PA}^\intercal\Sigma_P^\intercal\right)\tilde{\theta} \\
-\dfrac{1}{2}\left(\dfrac{1}{\gamma}-1\right)\tilde{\theta}^\intercal\Sigma_P\Sigma_P^\intercal\tilde{\theta}-\dfrac{1}{2}\mathrm{tr}\left(\mathrm{diag}\left(\tilde{\theta}\right)\Sigma_P\Sigma_P^\intercal\right)
\end{array}\right)
\end{array}\right) \\
&-\left(\mu_X^\intercal+\left(\frac{1}{\gamma}-1\right)\left((\mu_A-r\mathbb{1})^\intercal\left(\Sigma_A\Sigma_A^\intercal\right)^{-1}\Sigma_A\rho_{XA}^\intercal-\tilde{\theta}^\intercal\Sigma_P\rho_{XP}^\intercal\right)\Sigma_X^\intercal\right)\tilde{h}_X \\
&-\frac{1}{2}\mathrm{tr}\left(\tilde{h}_{XX^\intercal}\Sigma_X\Sigma_X^\intercal\right)
\end{aligned}
$$

with boundary condition $\tilde{h}(s,X_t;s)=1$ at any terminal date $s\in[t,T]$, and this other one for h

$$
\begin{aligned}
\frac{\partial h}{\partial t}=&h(t,X_t;T,\theta)\left(\begin{array}{l}
\dfrac{\delta-(1-\gamma)\left(r+\frac{1}{2}\frac{(\mu_A-r\mathbb{1})^\intercal\left(\Sigma_A\Sigma_A^\intercal\right)^{-1}(\mu_A-r\mathbb{1})}{\gamma}\right)}{\gamma} \\
+\left(\dfrac{1}{\gamma}-1\right)\left(\begin{array}{l}
\left(\mu_P^\intercal+\dfrac{1-\gamma}{\gamma}(\mu_A-r\mathbb{1})^\intercal\left(\Sigma_A\Sigma_A^\intercal\right)^{-1}\Sigma_A\rho_{PA}^\intercal\Sigma_P^\intercal\right)\theta \\
-\dfrac{1}{2}\left(\dfrac{1}{\gamma}-1\right)\theta^\intercal\Sigma_P\Sigma_P^\intercal\theta-\dfrac{1}{2}\mathrm{tr}\left(\mathrm{diag}\left(\theta\right)\Sigma_P\Sigma_P^\intercal\right)
\end{array}\right)
\end{array}\right) \\
&-\left(\mu_X^\intercal+\left(\frac{1}{\gamma}-1\right)\left((\mu_A-r\mathbb{1})^\intercal\left(\Sigma_A\Sigma_A^\intercal\right)^{-1}\Sigma_A\rho_{XA}^\intercal-\theta^\intercal\Sigma_P\rho_{XP}^\intercal\right)\Sigma_X^\intercal\right)h_X
\end{aligned}
$$

$$- \frac{1}{2} \operatorname{tr}\left(h_{XX^\intercal} \Sigma_X \Sigma_X^\intercal\right)$$

with boundary condition $h(T, X_t; T, \theta) = 1$. The PDEs for h and $\tilde{h}$ coincide up to the values of parameters θ and $\tilde{\theta}$, thus $\tilde{h}(t, X_t; T) = h(t, X_t; T, \tilde{\theta})$. In turn, $h(t, X_t; T, \theta)$ is a particular case of $h(t, X_t)$ from (38) in Proposition 2 under the restrictions of Assumption 2 and with parametrized consumption elasticity θ, inheriting its closed form solutions.

Solution to $\Upsilon(t, X_t, Q_t)$. The terms multiplying J_W in (A.29) give rise to a PDE for $\Upsilon(t, X_t, Q_t)$. Assuming that state indicators X_t, consumption prices P_t and payoffs Q_t are tradeable (Assumption 2), we have that

$$\begin{aligned}
0 =& \Sigma_X \left(I - \rho_{XA} \Sigma_A^\intercal \left(\Sigma_A \Sigma_A^\intercal\right)^{-1} \Sigma_A \rho_{XA}^\intercal\right) \Sigma_X^\intercal \\
0 =& \Sigma_P \left(\rho_{XP}^\intercal - \rho_{PA} \Sigma_A^\intercal \left(\Sigma_A \Sigma_A^\intercal\right)^{-1} \Sigma_A \rho_{XA}^\intercal\right) \Sigma_X^\intercal \\
0 =& \Sigma_X \left(\rho_{XQ} - \rho_{XA} \Sigma_A^\intercal \left(\Sigma_A \Sigma_A^\intercal\right)^{-1} \Sigma_A \rho_{QA}^\intercal\right) \Sigma_Q^\intercal \\
0 =& \Sigma_P \left(\rho_{PQ} - \rho_{PA} \Sigma_A^\intercal \left(\Sigma_A \Sigma_A^\intercal\right)^{-1} \Sigma_A \rho_{QA}^\intercal\right) \Sigma_Q^\intercal
\end{aligned}$$

and then the PDE for $\Upsilon(t, X_t, Q_t)$ simplifies to

$$\begin{aligned}
0 =& - \left(\Upsilon(t, X_t, Q_t) r + \left(\Upsilon_X^\intercal \Sigma_X \rho_{XA} + Q_t \Upsilon_Q \Sigma_Q \rho_{QA}\right)^\intercal \Sigma_A^\intercal \left(\Sigma_A \Sigma_A^\intercal\right)^{-1} \left(\mu_A - r\mathbb{1}\right) - Q_t \mathbb{1}_{t \le T_R}\right) \\
& + \begin{pmatrix} \dfrac{\partial \Upsilon}{\partial t} + \Upsilon_X^\intercal \mu_X + Q_t \Upsilon_Q \mu_Q + \dfrac{1}{2} \operatorname{tr}\left(\Upsilon_{XX^\intercal} \Sigma_X \Sigma_X^\intercal\right) + \dfrac{1}{2} \Upsilon_{QQ} Q_t \Sigma_Q \Sigma_Q^\intercal Q_t \\ + \Upsilon_{XQ}^\intercal \Sigma_X \rho_{XQ} \Sigma_Q^\intercal Q_t \end{pmatrix}
\end{aligned}$$

with boundary condition $\Upsilon(T, X_t, Q_t) = 0$.

The time derivative implied by this PDE has two stages, one for $t \le T_R$ and another for $t > T_R$

$$\frac{\partial \Upsilon}{\partial t} = \begin{cases}
\begin{pmatrix} \Upsilon(t, X_t, Q_t) r + \left(\Upsilon_X^\intercal \Sigma_X \rho_{XA} + Q_t \Upsilon_Q \Sigma_Q \rho_{QA}\right) \Sigma_A^\intercal \left(\Sigma_A \Sigma_A^\intercal\right)^{-1} \left(\mu_A - r\mathbb{1}\right) \\ - Q_t - \begin{pmatrix} \Upsilon_X^\intercal \mu_X + Q_t \Upsilon_Q \mu_Q + \dfrac{1}{2} \operatorname{tr}\left(\Upsilon_{XX^\intercal} \Sigma_X \Sigma_X^\intercal\right) + \dfrac{1}{2} \Upsilon_{QQ} Q_t \Sigma_Q \Sigma_Q^\intercal Q_t \\ + \Upsilon_{XQ}^\intercal \Sigma_X \rho_{XQ} \Sigma_Q^\intercal Q_t \end{pmatrix} \end{pmatrix} & \text{if } t \le T_R \\
\begin{pmatrix} \Upsilon(t, X_t, Q_t) r + \left(\Upsilon_X^\intercal \Sigma_X \rho_{XA} + Q_t \Upsilon_Q \Sigma_Q \rho_{QA}\right) \Sigma_A^\intercal \left(\Sigma_A \Sigma_A^\intercal\right)^{-1} \left(\mu_A - r\mathbb{1}\right) \\ - \begin{pmatrix} \Upsilon_X^\intercal \mu_X + Q_t \Upsilon_Q \mu_Q + \dfrac{1}{2} \operatorname{tr}\left(\Upsilon_{XX^\intercal} \Sigma_X \Sigma_X^\intercal\right) + \dfrac{1}{2} \Upsilon_{QQ} Q_t \Sigma_Q \Sigma_Q^\intercal Q_t \\ + \Upsilon_{XQ}^\intercal \Sigma_X \rho_{XQ} \Sigma_Q^\intercal Q_t \end{pmatrix} \end{pmatrix} & \text{if } t > T_R
\end{cases}$$

At time T the boundary condition $\Upsilon(T, X_t, Q_t) = 0$ implies that $\frac{\partial \Upsilon}{\partial t}\big|_{t=T} = 0$. By backwards induction we can see that this relation holds steady during the second stage $t \in (T_R, T]$, making Υ zero and in particular $\Upsilon(T_R, X_{T_R}, Q_{T_R}) = 0$. Thus $\Upsilon(t, X_t, Q_t)$ is the solution to the first stage PDE

$$\begin{aligned}
0 =& Q_t - \Upsilon(t, X_t, Q_t) r + \Upsilon_X^\intercal \left(\mu_X - \Sigma_X \rho_{XA} \Sigma_A^\intercal \left(\Sigma_A \Sigma_A^\intercal\right)^{-1} \left(\mu_A - r\mathbb{1}\right)\right) \\
& + Q_t \Upsilon_Q \left(\mu_Q - \Sigma_Q \rho_{QA} \Sigma_A^\intercal \left(\Sigma_A \Sigma_A^\intercal\right)^{-1} \left(\mu_A - r\mathbb{1}\right)\right) \\
& + \frac{\partial \Upsilon}{\partial t} + \frac{1}{2} \operatorname{tr}\left(\Upsilon_{XX^\intercal} \Sigma_X \Sigma_X^\intercal\right) + \frac{1}{2} \Upsilon_{QQ} Q_t \Sigma_Q \Sigma_Q^\intercal Q_t + \Upsilon_{XQ}^\intercal \Sigma_X \rho_{XQ} \Sigma_Q^\intercal Q_t
\end{aligned}$$

with boundary condition $\Upsilon(T_R, X_t, Q_t) = 0$ and it stays at zero thereafter.

This PDE coincides with the payoff stream price PDE (A.22) for $\Upsilon(t, X_t, Q_t; T_R)$ from Lemma 5 and inherits its closed form solutions. Note that the equivalence $Q_t \Upsilon_Q = \Upsilon(t, X_t, Q_t; T_R)$ from Lemma 5 can be replaced into (A.31), obtaining (46).

A.15 Proof of wealth ratios under extreme risk aversion, Remark 3

See Section A.15. First I will prove that, as $\gamma \to \infty$, the optimal bundles consumption rate is constant (49) through $\mathrm{d}v(c_t^\star, \tilde{\theta}, P_t) = 0$. The optimal bundle consumption rate can be obtained from Lemma 1 and then plugging the optimal consumption rate $c_t^\star$ from (45)

$$v(c_t^\star, \tilde{\theta}, P_t) = \frac{W_t + \Upsilon(t, X_t, Q_t; T_R)}{f(t, X_t, P_t)}$$

Dynamics of the optimal bundle consumption rate are derived using Itô's lemma

$$\begin{aligned}\mathrm{d}v(c_t^\star, \tilde{\theta}, P_t) =& \frac{\mathrm{d}W_t + \mathrm{d}\Upsilon(t, X_t, Q_t; T_R)}{f(t, X_t, P_t)} - \frac{W_t + \Upsilon(t, X_t, Q_t; T_R)}{f(t, X_t, P_t)^2} \mathrm{d}f(t, X_t, P_t) \\ &+ \frac{W_t + \Upsilon(t, X_t, Q_t; T_R)}{f(t, X_t, P_t)^3} \mathrm{d}[f(t, X_t, P_t)] \\ &- \frac{\mathrm{d}[W_t, f(t, X_t, P_t)] + \mathrm{d}[\Upsilon(t, X_t, Q_t; T_R), f(t, X_t, P_t)]}{f(t, X_t, P_t)^2}\end{aligned}$$

where $\mathrm{d}[\cdot]$ denotes the quadratic variation and $\mathrm{d}[\cdot, \cdot]$ the quadratic covariation. Note that, in what follows, (2) is used to simplify and cancel out terms. Plugging wealth dynamics from (32) and using Itô's lemma for the dynamics of $\Upsilon(t, X_t, Q_t; T_R)$ and $f(t, X_t, P_t)$, yields

$$\begin{aligned}\mathrm{d}v(c_t^\star, \tilde{\theta}, P_t) =& \frac{(Q_t \mathbb{1}_{t \le T_R} - c_t^\star)\,\mathrm{d}t + W_t r\,\mathrm{d}t + W_t \pi^\intercal\left((\mu_A - r\mathbb{1})\,\mathrm{d}t + \Sigma_A\,\mathrm{d}Z_{A,t}\right)}{f(t, X_t, P_t)} \\ &+ \frac{\frac{\partial \Upsilon}{\partial t}\,\mathrm{d}t + \Upsilon_X\,\mathrm{d}X_t + \Upsilon_Q\,\mathrm{d}Q_t + \frac{1}{2}\Upsilon_{XX}\,\mathrm{d}[X_t, X_t] + \frac{1}{2}\Upsilon_{QQ}\,\mathrm{d}[Q_t, Q_t] + \Upsilon_{XQ}\,\mathrm{d}[X_t, Q_t]}{f(t, X_t, P_t)} \\ &- \frac{W_t + \Upsilon(t, X_t, Q_t; T_R)}{f(t, X_t, P_t)^2}\begin{pmatrix}\frac{\partial f}{\partial t}\,\mathrm{d}t + f_X\,\mathrm{d}X_t + f_P\,\mathrm{d}P_t + \frac{1}{2} f_{XX}\,\mathrm{d}[X_t, X_t] \\ + \frac{1}{2} f_{PP}\,\mathrm{d}[P_t, P_t] + f_{XP}\,\mathrm{d}[X_t, P_t] - \frac{\mathrm{d}[f(t, X_t, P_t)]}{f(t, X_t, P_t)}\end{pmatrix} \\ &- \frac{\mathrm{d}[W_t, f(t, X_t, P_t)] + \mathrm{d}[\Upsilon(t, X_t, Q_t; T_R), f(t, X_t, P_t)]}{f(t, X_t, P_t)^2}\end{aligned}$$

then expanding $c_t^\star$ with (45) and quadratic covariation terms related to W_t, $\Upsilon(t, X_t, Q_t; T_R)$ and $f(t, X_t, P_t)$ yields

$$\begin{aligned}\mathrm{d}v(c_t^\star, \tilde{\theta}, P_t) =& \frac{W_t \pi^{\star\intercal}\left((\mu_A - r\mathbb{1})\,\mathrm{d}t + \Sigma_A\,\mathrm{d}Z_{A,t} - \Sigma_A\left(\rho_{XA}^\intercal \Sigma_X^\intercal \frac{f_X^\intercal}{f(t,X_t,P_t)} + \rho_{PA}^\intercal \Sigma_P^\intercal \operatorname{diag}(P_t) \frac{f_P^\intercal}{f(t,X_t,P_t)}\right)\mathrm{d}t\right)}{f(t, X_t, P_t)} \\ &+ \frac{\begin{pmatrix}\frac{\partial \Upsilon}{\partial t}\,\mathrm{d}t + Q_t \mathbb{1}_{t \le T_R}\,\mathrm{d}t + \Upsilon_X\,\mathrm{d}X_t + \Upsilon_Q\,\mathrm{d}Q_t + \frac{1}{2}\Upsilon_{XX}\,\mathrm{d}[X_t, X_t] \\ + \frac{1}{2}\Upsilon_{QQ}\,\mathrm{d}[Q_t, Q_t] + \Upsilon_{XQ}\,\mathrm{d}[X_t, Q_t]\end{pmatrix}}{f(t, X_t, P_t)}\end{aligned}$$

$$
\begin{aligned}
&- \frac{W_t + \Upsilon(t, X_t, Q_t; T_R)}{f(t, X_t, P_t)^2} \begin{pmatrix} \dfrac{\partial f}{\partial t}\,\mathrm{d}t + f_X\,\mathrm{d}X_t + f_P\,\mathrm{d}P_t + \dfrac{1}{2} f_{XX}\,\mathrm{d}[X_t, X_t] \\ + \dfrac{1}{2} f_{PP}\,\mathrm{d}[P_t, P_t] + f_{XP}\,\mathrm{d}[X_t, P_t] - \dfrac{\mathrm{d}[f(t, X_t, P_t)]}{f(t, X_t, P_t)} \\ + \left(e^{(1-\tilde{\theta}^\intercal \mathbb{1})\log\left(1-\tilde{\theta}^\intercal \mathbb{1}\right) + \tilde{\theta}^\intercal \log\left(\tilde{\theta}\right) - \tilde{\theta}^\intercal \log(P_t)}\right)^{-1} \mathrm{d}t \end{pmatrix} \\
&+ \frac{W_t r\,\mathrm{d}t}{f(t, X_t, P_t)} - \frac{\begin{pmatrix} \Upsilon_X \Sigma_X \Sigma_X^\intercal f_X\,\mathrm{d}t + \Upsilon_X \Sigma_X \rho_{XP} \Sigma_P^\intercal \operatorname{diag}(P_t) f_P\,\mathrm{d}t \\ + \Upsilon_Q Q_t \Sigma_Q \rho_{XQ}^\intercal \Sigma_X^\intercal f_X\,\mathrm{d}t + \Upsilon_Q Q_t \Sigma_Q \rho_{PQ}^\intercal \Sigma_P^\intercal \operatorname{diag}(P_t) f_P\,\mathrm{d}t \end{pmatrix}}{f(t, X_t, P_t)^2}
\end{aligned}
$$

The next step is to replace $\pi^\star$ with (46) and make some cancellations

$$
\begin{aligned}
\mathrm{d}v(c_t^\star, \tilde{\theta}, P_t) =& \frac{(W_t + \Upsilon(t, X_t, Q_t; T_R))}{f(t, X_t, P_t)^2} \begin{pmatrix} \rho_{XA}^\intercal \Sigma_X^\intercal f_X \\ + \rho_{PA}^\intercal \Sigma_P^\intercal \operatorname{diag}(P_t) f_P \end{pmatrix}^\intercal \left(\Sigma_A^{-1}(\mu_A - r\mathbb{1})\,\mathrm{d}t + \mathrm{d}Z_{A,t}\right) \\
&- \frac{\left(\rho_{XA}^\intercal \Sigma_X^\intercal \Upsilon_X + \rho_{QA}^\intercal \Sigma_Q^\intercal Q_t \Upsilon_Q\right)^\intercal}{f(t, X_t, P_t)} \left(\Sigma_A^{-1}(\mu_A - r\mathbb{1})\,\mathrm{d}t + \mathrm{d}Z_{A,t}\right) \\
&- \frac{W_t + \Upsilon(t, X_t, Q_t; T_R)}{f(t, X_t, P_t)^3} \begin{pmatrix} \rho_{XA}^\intercal \Sigma_X^\intercal f_X \\ + \rho_{PA}^\intercal \Sigma_P^\intercal \operatorname{diag}(P_t) f_P \end{pmatrix}^\intercal \begin{pmatrix} \rho_{XA}^\intercal \Sigma_X^\intercal f_X^\intercal\,\mathrm{d}t \\ + \rho_{PA}^\intercal \Sigma_P^\intercal \operatorname{diag}(P_t) f_P^\intercal\,\mathrm{d}t \end{pmatrix} \\
&+ \frac{\begin{pmatrix} \dfrac{\partial \Upsilon}{\partial t}\,\mathrm{d}t + Q_t \mathbb{1}_{t \leq T_R}\,\mathrm{d}t + \Upsilon_X\,\mathrm{d}X_t + \Upsilon_Q\,\mathrm{d}Q_t + \dfrac{1}{2}\Upsilon_{XX}\,\mathrm{d}[X_t, X_t] \\ + \dfrac{1}{2}\Upsilon_{QQ}\,\mathrm{d}[Q_t, Q_t] + \Upsilon_{XQ}\,\mathrm{d}[X_t, Q_t] \end{pmatrix}}{f(t, X_t, P_t)} \\
&- \frac{W_t + \Upsilon(t, X_t, Q_t; T_R)}{f(t, X_t, P_t)^2} \begin{pmatrix} \dfrac{\partial f}{\partial t}\,\mathrm{d}t + f_X\,\mathrm{d}X_t + f_P\,\mathrm{d}P_t + \dfrac{1}{2} f_{XX}\,\mathrm{d}[X_t, X_t] \\ + \dfrac{1}{2} f_{PP}\,\mathrm{d}[P_t, P_t] + f_{XP}\,\mathrm{d}[X_t, P_t] - \dfrac{\mathrm{d}[f(t, X_t, P_t)]}{f(t, X_t, P_t)} \\ + \left(e^{(1-\tilde{\theta}^\intercal \mathbb{1})\log\left(1-\tilde{\theta}^\intercal \mathbb{1}\right) + \tilde{\theta}^\intercal \log\left(\tilde{\theta}\right) - \tilde{\theta}^\intercal \log(P_t)}\right)^{-1} \mathrm{d}t \end{pmatrix} \\
&+ \frac{W_t r\,\mathrm{d}t}{f(t, X_t, P_t)}
\end{aligned}
$$

Then we need to expand $\mathrm{d}X_t = \frac{\partial X}{\partial t}\,\mathrm{d}t + \Sigma_X \rho_{XA}\,\mathrm{d}Z_{A,t}$, $\mathrm{d}Q_t = \frac{\partial Q}{\partial t}\,\mathrm{d}t + Q_t \Sigma_Q \rho_{QA}\,\mathrm{d}Z_{A,t}$ and $\mathrm{d}P_t = \frac{\partial P}{\partial t}\,\mathrm{d}t + \operatorname{diag}(P_t)\Sigma_P \rho_{PA}\,\mathrm{d}Z_{A,t}$ terms to cancel out against $\mathrm{d}Z_{A,t}$, expand $\mathrm{d}[f(t, X_t, P_t)]$ to cancel out with matching terms and separate $W_t r = (W_t + \Upsilon(t, X_t, Q_t; T_R))r - \Upsilon(t, X_t, Q_t; T_R)r$

to arrive at

$$
\mathrm{d}v(c_t^\star,\tilde{\theta},P_t) = \frac{\begin{pmatrix} + \dfrac{\partial \Upsilon}{\partial t}\,\mathrm{d}t \\ - \Upsilon(t,X_t,Q_t;T_R) r\,\mathrm{d}t \\ + Q_t \mathbb{1}_{t\le T_R}\,\mathrm{d}t \\ + \Upsilon_X \left(\dfrac{\partial X}{\partial t} - (\rho_{XA}^\intercal \Sigma_X^\intercal)^\intercal \Sigma_A^{-1}(\mu_A - r\mathbb{1})\right)\mathrm{d}t \\ + \Upsilon_Q \left(\dfrac{\partial Q}{\partial t} - \left(\rho_{QA}^\intercal \Sigma_Q^\intercal Q_t\right)^\intercal \Sigma_A^{-1}(\mu_A - r\mathbb{1})\right)\mathrm{d}t \\ + \dfrac{1}{2}\Upsilon_{XX}\,\mathrm{d}[X_t,X_t] \\ + \dfrac{1}{2}\Upsilon_{QQ}\,\mathrm{d}[Q_t,Q_t] \\ + \Upsilon_{XQ}\,\mathrm{d}[X_t,Q_t] \end{pmatrix}}{f(t,X_t,P_t)}
- \frac{W_t + \Upsilon(t,X_t,Q_t;T_R)}{f(t,X_t,P_t)^2}\begin{pmatrix} + \dfrac{\partial f}{\partial t}\,\mathrm{d}t \\ + \left(e^{(1-\tilde{\theta}^\intercal \mathbb{1})\log\left(1-\tilde{\theta}^\intercal\mathbb{1}\right)+\tilde{\theta}^\intercal \log\left(\tilde{\theta}\right)-\tilde{\theta}^\intercal \log(P_t)}\right)^{-1}\mathrm{d}t \\ - r f(t,X_t,P_t)\,\mathrm{d}t \\ + f_X\left(\dfrac{\partial X}{\partial t} - (\rho_{XA}^\intercal \Sigma_X^\intercal)^\intercal \Sigma_A^{-1}(\mu_A - r\mathbb{1})\right)\mathrm{d}t \\ + f_P\left(\dfrac{\partial P}{\partial t} - (\rho_{PA}^\intercal \Sigma_P^\intercal \operatorname{diag}(P_t))^\intercal \Sigma_A^{-1}(\mu_A - r\mathbb{1})\right)\mathrm{d}t \\ + \dfrac{1}{2} f_{XX}\,\mathrm{d}[X_t,X_t] \\ + \dfrac{1}{2} f_{PP}\,\mathrm{d}[P_t,P_t] \\ + f_{XP}\,\mathrm{d}[X_t,P_t] \end{pmatrix}
$$

The two expressions in parentheses are zero due to PDE conditions (A.32) and (A.22) under the previously stated assumptions, thus

$$
\mathrm{d}v(c_t^\star,\tilde{\theta},P_t) = 0.
$$

The remaining statements assume that intermediate and terminal product elasticities coincide $\tilde{\theta} = \theta$ and the optimal bundle price $P_{v,t} = \left(e^{(1-\theta^\intercal\mathbb{1})\log(1-\theta^\intercal\mathbb{1})+\theta^\intercal\log(\theta)-\theta^\intercal\log(P_t)}\right)^{-1}$ is related to the payoff process Q_t through a function that only depends on state X_t. Without loss of generality we can model this relationship throught the log-wedge function $\nu_v(X_t)$ such that $\frac{P_{v,t}}{Q_t} = e^{\nu_v(X_t)}$. Solving for wealth in (49) results in

$$
W_t = \frac{f(t,X_t,P_t)}{f(t_0,X_{t_0},P_{t_0})}\left(W_{t_0} + \Upsilon(t_0,X_{t_0},Q_{t_0};T_R)\right) - \Upsilon(t,X_t,Q_t;T_R)
$$

which under the previous assumptions yields the following ratios

$$
\frac{W_t}{P_{v,t}} = \frac{\left(\mathbb{1}_{\varepsilon_2\neq 0} h(t,X_t;T,\theta) + \mathbb{1}_{\varepsilon_1\neq 0}\int_t^T h(t,X_t;s,\theta)\,\mathrm{d}s\right)}{\left(\mathbb{1}_{\varepsilon_2\neq 0} h(t_0,X_{t_0};T,\theta) + \mathbb{1}_{\varepsilon_1\neq 0}\int_{t_0}^T h(t_0,X_{t_0};s,\theta)\,\mathrm{d}s\right)}\left(\frac{W_{t_0}}{P_{v,t_0}} + \frac{\int_{t_0}^{T_R}\tilde{\Omega}(t_0,X_{t_0};s)\,\mathrm{d}s}{e^{\nu_v(X_{t_0})}}\right)
$$

$$- \frac{\int_t^{T_R} \tilde{\Omega}(t, X_t; s)\,\mathrm{d}s}{e^{\nu_v(X_t)}} \tag{A.42}$$

$$\begin{aligned}\frac{W_t}{Q_t} =& e^{\nu_v(X_t)-\nu_v(X_{t_0})} \frac{\left(\mathbb{1}_{\varepsilon_2 \neq 0} h(t, X_t; T, \theta) + \mathbb{1}_{\varepsilon_1 \neq 0} \int_t^T h(t, X_t; s, \theta)\,\mathrm{d}s\right)}{\left(\mathbb{1}_{\varepsilon_2 \neq 0} h(t_0, X_{t_0}; T, \theta) + \mathbb{1}_{\varepsilon_1 \neq 0} \int_{t_0}^T h(t_0, X_{t_0}; s, \theta)\,\mathrm{d}s\right)} \\ &\cdot \left(\frac{W_{t_0}}{Q_{t_0}} + \int_{t_0}^{T_R} \tilde{\Omega}(t_0, X_{t_0}; s)\,\mathrm{d}s\right) - \int_t^{T_R} \tilde{\Omega}(t, X_t; s)\,\mathrm{d}s \end{aligned} \tag{A.43}$$

$$\begin{aligned}\frac{W_t}{f(t, X_t, P_t)} =& \frac{W_{t_0}}{f(t_0, X_{t_0}, P_{t_0})} + \frac{e^{-\nu_v(X_{t_0})} \int_{t_0}^{T_R} \tilde{\Omega}(t_0, X_{t_0}; s)\,\mathrm{d}s}{\left(\mathbb{1}_{\varepsilon_2 \neq 0} h(t_0, X_{t_0}; T, \theta) + \mathbb{1}_{\varepsilon_1 \neq 0} \int_{t_0}^T h(t_0, X_{t_0}; s, \theta)\,\mathrm{d}s\right)} \\ &- \frac{e^{-\nu_v(X_t)} \int_t^{T_R} \tilde{\Omega}(t, X_t; s)\,\mathrm{d}s}{\left(\mathbb{1}_{\varepsilon_2 \neq 0} h(t, X_t; T, \theta) + \mathbb{1}_{\varepsilon_1 \neq 0} \int_t^T h(t, X_t; s, \theta)\,\mathrm{d}s\right)} \end{aligned} \tag{A.44}$$

$$\begin{aligned}\frac{W_t}{\Upsilon(t, X_t, Q_t; T_R)} =& e^{\nu_v(X_t)-\nu_v(X_{t_0})} \frac{\left(\mathbb{1}_{\varepsilon_2 \neq 0} h(t, X_t; T, \theta) + \mathbb{1}_{\varepsilon_1 \neq 0} \int_t^T h(t, X_t; s, \theta)\,\mathrm{d}s\right)}{\left(\mathbb{1}_{\varepsilon_2 \neq 0} h(t_0, X_{t_0}; T, \theta) + \mathbb{1}_{\varepsilon_1 \neq 0} \int_{t_0}^T h(t_0, X_{t_0}; s, \theta)\,\mathrm{d}s\right)} \\ &\cdot \frac{\int_{t_0}^{T_R} \tilde{\Omega}(t_0, X_{t_0}; s)\,\mathrm{d}s}{\int_t^{T_R} \tilde{\Omega}(t, X_t; s)\,\mathrm{d}s} \left(\frac{W_{t_0}}{\Upsilon(t_0, X_{t_0}, Q_{t_0}; T_R)} + 1\right) - 1 \end{aligned} \tag{A.45}$$

Statement (i.) holds because the ratios (A.42), (A.43), (A.44) and (A.45) do not depend on Q_t or P_t, they only depend on some time references and the state X_t.

Statement (ii.) refers to stationarity in the sense that the unconditional joint distribution of process elements is independent of time, that is, the distribution is invariant to time shifts. Assuming that market parameters are Markovian with respect to state process X_t, then $\tilde{\Omega}$ and h dynamics only depend on X_t. The only time references used to describe $\tilde{\Omega}$ and h are the initial t and terminal s times of the process. Neither dynamics nor boundary values depend on time; the initial and terminal time points only affect the time horizon over which $\tilde{\Omega}$ and h are defined. The PDEs characterizing $\tilde{\Omega}$ and h have deterministic parameters and describe deterministic functions of inputs X_t and $T - t$. Even though integrands may depend on inputs X_t and $s - t$, the integral as a whole can be defined as a function of X_t and $T - t$. Keeping the time horizon $T - t$ constant in (A.42), (A.43), (A.44) and (A.45) through partial function application yields mapping functions that only depend on X_t. As X_t is assumed to be jointly stationary, this makes the mapped process stationary as well. A similar argument applies when assuming that the reference horizon $T - t$ is jointly stationary with state. Note that so far I implicitly assumed that the mapping function is measurable since this condition makes it possible to obtain a well-defined pushforward measure (Kallenberg, 2021, Lemma 25.1) in this context.

Statement (iii.) clearly holds as removing the dependency on state X_t from (A.42), (A.43), (A.44) and (A.45) makes them deterministic and depend only on time.

The remaining statements assume further that the optimal bundle price $P_{v,t}$ is a constant proportion of the payoff process Q_t at all times, $\frac{P_{v,t}}{Q_t} = e^{\bar{\nu}_v}$. Let me now establish some auxiliary relations that will be useful during the analysis. These assumptions imply that $\mathrm{d}\log(P_{v,t}) = \mathrm{d}\log(Q_t)$ under any possible vector price P_t,

$$\mathrm{d}\log(Q_t) = \mathrm{d}\theta^\intercal \log(P_t)$$

$$\mu_Q \, \mathrm{d}t + \Sigma_Q \rho_{QA} \, \mathrm{d}Z_{A,t} - \frac{1}{2}\Sigma_Q \Sigma_Q^\intercal \, \mathrm{d}t = \theta^\intercal \left(\mu_P \, \mathrm{d}t + \Sigma_P \rho_{PA} \, \mathrm{d}Z_{A,t} \right) - \frac{1}{2} \operatorname{tr}\left(\operatorname{diag}(\theta) \Sigma_P \Sigma_P^\intercal \right) \mathrm{d}t$$

which means that risk exposures and drift terms should match.

$$\Sigma_Q \rho_{QA} = \theta^\intercal \Sigma_P \rho_{PA} \tag{A.46}$$

$$\mu_Q - \frac{1}{2}\Sigma_Q \Sigma_Q^\intercal = \theta^\intercal \mu_P - \frac{1}{2} \operatorname{tr}\left(\operatorname{diag}(\theta) \Sigma_P \Sigma_P^\intercal \right) \tag{A.47}$$

Combining both restrictions we arrive at

$$\mu_Q = \theta^\intercal \mu_P + \frac{1}{2} \theta^\intercal \Sigma_P \Sigma_P^\intercal \theta - \frac{1}{2} \operatorname{tr}\left(\operatorname{diag}(\theta) \Sigma_P \Sigma_P^\intercal \right). \tag{A.48}$$

In turn we have that $h(t, X_t; T, \theta) = \tilde{\Omega}(t, X_t; T)$ since the boundaries and dynamics of PDEs (13) and (38) coincide under the stated assumptions. Note that plugging (A.48) and (A.46) into (38) can help to see this equivalence.

Regarding statement (iv.), if $\varepsilon_1 = 1, \varepsilon_2 = 0$ and $T_R = T$ we have that the following ratios are constant

$$\frac{W_t}{f(t, X_t, P_t)} = \frac{W_{t_0} + \Upsilon(t_0, X_{t_0}, Q_{t_0}; T)}{f(t_0, X_{t_0}, P_{t_0})} - e^{-\bar{\nu}_v}$$

$$\frac{W_t}{\Upsilon(t, X_t, Q_t; T_R)} = e^{\bar{\nu}_v} \frac{W_{t_0} + \Upsilon(t_0, X_{t_0}, Q_{t_0}; T)}{f(t_0, X_{t_0}, P_{t_0})} - 1.$$

where I used $h(t, X_t; T, \theta) = \tilde{\Omega}(t, X_t; T)$ derived earlier earlier. Additionally if $W_{t_0} = 0$ then (A.42) and (A.43) reduce to

$$\frac{W_t}{P_{v,t}} = e^{-\bar{\nu}_v} \left(\int_t^T h(t, X_t; s, \theta) \, \mathrm{d}s \, \frac{\int_{t_0}^T \tilde{\Omega}(t_0, X_{t_0}; s) \, \mathrm{d}s}{\int_{t_0}^T h(t_0, X_{t_0}; s, \theta) \, \mathrm{d}s} - \int_t^T \tilde{\Omega}(t, X_t; s) \, \mathrm{d}s \right) = 0$$

$$\frac{W_t}{Q_t} = \int_t^T h(t, X_t; s, \theta) \, \mathrm{d}s \, \frac{\int_{t_0}^T \tilde{\Omega}(t_0, X_{t_0}; s) \, \mathrm{d}s}{\int_{t_0}^T h(t_0, X_{t_0}; s, \theta) \, \mathrm{d}s} - \int_t^T \tilde{\Omega}(t, X_t; s) \, \mathrm{d}s = 0$$

so wealth stays at zero at all times when initial wealth is zero.

Lastly, showing the equivalence $f(t, X_t, P_t) = e^{\bar{\nu}_v} \Upsilon(t, X_t, Q_t; T)$ when there is only intermediate consumption $\varepsilon_1 = 1, \varepsilon_2 = 0$ is straightforward using the fact that $h(t, X_t; T, \theta) = \tilde{\Omega}(t, X_t; T)$

$$f(t, X_t, P_t) = P_{v,t} \int_t^T h(t, X_t; s, \theta) \, \mathrm{d}s = P_{v,t} \int_t^T \tilde{\Omega}(t, X_t; s) \, \mathrm{d}s = e^{\bar{\nu}_v} \Upsilon(t, X_t, Q_t; T).$$

A.16 Proof of strategy equivalent wealth, Lemma 8

Because of strict monotonicity on initial wealth W_t, expected utility can replace initial wealth in the minimization objective (50)

$$\mathrm{EW}(\pi, \bar{\pi}, W_t) = \arg\min_{\tilde{W}} U(\bar{\pi}, \tilde{W})$$

$$\text{s.t. } U(\bar{\pi}, \tilde{W}) \geq U(\pi, W_t).$$

The range of U does not depend on the strategy and the function is continuous on initial wealth, so there exists a $\tilde{W}$ such that $U(\bar{\pi}, \tilde{W}) = U(\pi, W_t)$. This value coincides with the minimum to the problem above and strict monotonicity makes the function invertible, thus the strategy equivalent wealth reduces to

$$\mathrm{EW}(\pi, \bar{\pi}, W_t) = \left(U(\bar{\pi}, \cdot)\right)^{-1} \left(U(\pi, W_t)\right).$$

A.17 Proof of expected utility with incomplete markets, Lemma 9

The Hamilton-Jacobi-Bellman equation can be derived following the steps of Section A.12 up to (A.27). With $\varepsilon_1 = 0$, $\varepsilon_2 = 1$, $Q_t = 0$, $c_t = 0$ and for any arbitrary but finite π, we arrive at

$$\begin{aligned}
0 =& U_W W_t \pi^\intercal (\mu_A - r\mathbb{1}) \\
&+ \frac{1}{2} U_{WW} W_t^2 \pi^\intercal \Sigma_A \Sigma_A^\intercal \pi + U_{WX}^\intercal \Sigma_X \rho_{XA} \Sigma_A^\intercal \pi W_t + U_{WP}^\intercal \operatorname{diag}(P_t) \Sigma_P \rho_{PA} \Sigma_A^\intercal \pi W_t \\
&- \delta U(t, W_t, X_t, P_t; \pi) + \frac{\partial U}{\partial t} + U_W W_t r + U_X^\intercal \mu_X + \frac{1}{2} \operatorname{tr}\left(U_{XX^\intercal} \Sigma_X \Sigma_X^\intercal\right) \\
&+ U_P^\intercal \operatorname{diag}(P_t) \mu_P + \frac{1}{2} \operatorname{tr}\left(U_{PP^\intercal} \operatorname{diag}(P_t) \Sigma_P \Sigma_P^\intercal \operatorname{diag}(P_t)\right) + \operatorname{tr}\left(U_{XP^\intercal}^\intercal \Sigma_X \rho_{XP} \Sigma_P^\intercal \operatorname{diag}(P_t)\right)
\end{aligned}$$

with boundary condition $U(T, W_t, X_t, P_t; \pi) = \left(e^{(1-\theta^\intercal\mathbb{1})\log(1-\theta^\intercal\mathbb{1}) + \theta^\intercal \log(\theta) - \theta^\intercal \log(P_t)}\right)^{1-\gamma} \frac{W_T^{1-\gamma}}{1-\gamma}$.

Consider the following ansatz

$$U(t, W_t, X_t, P_t; \pi) = \frac{W_t^{1-\gamma}}{1-\gamma} f(t, X_t, P_t; \pi)^\gamma$$

then

$$\begin{aligned}
0 =& \frac{\partial f}{\partial t} \\
&- f(t, X_t, P_t; \pi) \frac{\delta - (1-\gamma)\left(r + \pi^\intercal(\mu_A - r\mathbb{1}) - \gamma \frac{1}{2} \pi^\intercal \Sigma_A \Sigma_A^\intercal \pi\right)}{\gamma} \\
&+ f_X^\intercal \left(\mu_X + (1-\gamma) \Sigma_X \rho_{XA} \Sigma_A^\intercal \pi\right) \\
&+ f(t, X_t, P_t; \pi) \frac{1}{2} \operatorname{tr}\left(\left((\gamma - 1) f(t, X_t, P_t; \pi)^{-2} f_X f_X^\intercal + f(t, X_t, P_t; \pi)^{-1} f_{XX^\intercal}\right) \Sigma_X \Sigma_X^\intercal\right) \\
&+ f_P^\intercal \left(\operatorname{diag}(P_t) \mu_P + (1-\gamma) \operatorname{diag}(P_t) \Sigma_P \rho_{PA} \Sigma_A^\intercal \pi\right) \\
&+ \frac{1}{2} \operatorname{tr}\left(\left((\gamma - 1) f(t, X_t, P_t; \pi)^{-1} f_P f_P^\intercal + f_{PP^\intercal}\right) \operatorname{diag}(P_t) \Sigma_P \Sigma_P^\intercal \operatorname{diag}(P_t)\right) \\
&+ \operatorname{tr}\left(\left((\gamma - 1) f(t, X_t, P_t; \pi)^{-1} f_X f_P^\intercal + f_{XP^\intercal}\right)^\intercal \Sigma_X \rho_{XP} \Sigma_P^\intercal \operatorname{diag}(P_t)\right)
\end{aligned}$$

with boundary condition $f(T, X_t, P_t; \pi) = \left(e^{(1-\theta^\intercal\mathbb{1})\log(1-\theta^\intercal\mathbb{1}) + \theta^\intercal \log(\theta) - \theta^\intercal \log(P_t)}\right)^{\frac{1}{\gamma} - 1}$.

Using the following ansatz

$$f(t, X_t, P_t; \pi) = \left(e^{(1-\theta^\intercal\mathbb{1})\log(1-\theta^\intercal\mathbb{1}) + \theta^\intercal \log(\theta) - \theta^\intercal \log(P_t)}\right)^{\frac{1}{\gamma} - 1} h(t, X_t; T, \theta, \pi)$$

then

$$\frac{\partial h}{\partial t} = h(t, X_t; T, \theta, \pi)\left(\frac{\delta}{\gamma} - \left(\frac{1}{\gamma} - 1\right)\left(\begin{array}{l} r + (\mu_A - r\mathbb{1})^\intercal \pi - \frac{1}{2}\gamma \pi^\intercal \Sigma_A \Sigma_A^\intercal \pi \\ -\theta^\intercal\left(\mu_P + (1-\gamma)\Sigma_P \rho_{PA} \Sigma_A^\intercal \pi\right) \\ +\frac{1}{2}(1-\gamma)\theta^\intercal \Sigma_P \Sigma_P^\intercal \theta + \frac{1}{2}\operatorname{tr}\left(\operatorname{diag}(\theta)\Sigma_P \Sigma_P^\intercal\right) \end{array}\right)\right)$$
$$- h_X^\intercal\left(\mu_X + (1-\gamma)\Sigma_X\left(\rho_{XA}\Sigma_A^\intercal \pi - \rho_{XP}\Sigma_P^\intercal \theta\right)\right)$$
$$+ \frac{1}{2}(1-\gamma)\frac{1}{h(t, X_t; T, \theta, \pi)} h_X^\intercal \Sigma_X \Sigma_X^\intercal h_X - \frac{1}{2}\operatorname{tr}\left(h_{XX^\intercal}\Sigma_X \Sigma_X^\intercal\right) \tag{A.49}$$

with boundary condition $h(T, X_t; T, \theta, \pi) = 1$.

The PDE for $h(t, X_t; T, \theta, \pi)$ is a particular case of the generic PDE from Lemma 3 parametrized as

$$g\left(\begin{array}{l} t, X_t; \\ R = -\frac{\delta}{\gamma} + \left(\frac{1}{\gamma} - 1\right)\left(\begin{array}{l} r + (\mu_A - r\mathbb{1})^\intercal \pi - \frac{1}{2}\gamma \pi^\intercal \Sigma_A \Sigma_A^\intercal \pi - \theta^\intercal\left(\mu_P + (1-\gamma)\Sigma_P \rho_{PA}\Sigma_A^\intercal \pi\right) \\ +\frac{1}{2}(1-\gamma)\theta^\intercal \Sigma_P \Sigma_P^\intercal \theta + \frac{1}{2}\operatorname{tr}\left(\operatorname{diag}(\theta)\Sigma_P \Sigma_P^\intercal\right) \end{array}\right), \\ B = \mu_X + (1-\gamma)\Sigma_X\left(\rho_{XA}\Sigma_A^\intercal \pi - \rho_{XP}\Sigma_P^\intercal \theta\right), \\ C = \gamma \Sigma_X \Sigma_X^\intercal, \\ D = \Sigma_X \Sigma_X^\intercal \end{array}\right)$$

with boundary condition $g(T, X_t) = 1$.

Section A.5 explains how to reduce the generic PDE to a system of Riccati ODEs by parametrizing A, B, C, D quadratically, which in this case can be constructed from the following building blocks

$$\begin{aligned}
\delta &= {}_\delta\alpha + {}_\delta\beta_p\, X^p + X_p\, {\eta_h}^p\, {}_\delta{\omega^h}_m\, {\eta^m}_q\, X^q \\
r &= {}_r\alpha + {}_r\beta_p\, X^p + X_p\, {\eta_h}^p\, {}_r{\omega^h}_m\, {\eta^m}_q\, X^q \\
(\mu_A - r\mathbb{1})^\intercal \pi &= {}_A\alpha + {}_A\beta_p\, X^p + X_p\, {\eta_h}^p\, {}_A{\omega^h}_m\, {\eta^m}_q\, X^q \\
\pi^\intercal \Sigma_A \Sigma_A^\intercal \pi &= {}_{\Sigma_A}\alpha + {}_{\Sigma_A}\beta_p\, X^p + X_p\, {\eta_h}^p\, {}_{\Sigma_A}{\omega^h}_m\, {\eta^m}_q\, X^q \\
(\mu_P)^k &= {}_P\alpha^k + {}_P{\beta^k}_p\, X^p + X_p\, {\eta_h}^p\, {}_P{\omega^{kh}}_m\, {\eta^m}_q\, X^q \\
(\Sigma_P \rho_{PA}\Sigma_A^\intercal \pi)^k &= {}_{\Sigma_{PA}}\alpha^k + {}_{\Sigma_{PA}}{\beta^k}_p\, X^p + X_p\, {\eta_h}^p\, {}_{\Sigma_{PA}}{\omega^{kh}}_m\, {\eta^m}_q\, X^q \\
{(\Sigma_P \Sigma_P^\intercal)^k}_l &= {}_{\Sigma_P}{\alpha^k}_l + {}_{\Sigma_P}{\beta^k}_{lp}\, X^p + X_p\, {\eta_h}^p\, {}_{\Sigma_P}{\omega^k}_l{}^h{}_m\, {\eta^m}_q\, X^q \\
(\mu_X)^k &= {}_X\alpha^k - {}_X{\beta^k}_p\, X^p + X_p\, {\eta_h}^p\, {}_X{\omega^{kh}}_m\, {\eta^m}_q\, X^q \\
(\Sigma_X \rho_{XA}\Sigma_A^\intercal \pi)^k &= {}_{\Sigma_{XA}}\alpha^k + {}_{\Sigma_{XA}}{\beta^k}_p\, X^p + X_p\, {\eta_h}^p\, {}_{\Sigma_{XA}}{\omega^{kh}}_m\, {\eta^m}_q\, X^q \\
{(\Sigma_X \rho_{XP}\Sigma_P^\intercal)^k}_l &= {}_{\Sigma_X\Sigma_P}{\alpha^k}_l + {}_{\Sigma_X\Sigma_P}{\beta^k}_{lp}\, X^p + X_p\, {\eta_h}^p\, {}_{\Sigma_X\Sigma_P}{\omega^k}_l{}^h{}_m\, {\eta^m}_q\, X^q \\
{(\Sigma_X \Sigma_X^\intercal)^k}_l &= {}_{\Sigma_X}{\alpha^k}_l + {}_{\Sigma_X}{\beta^k}_{lp}\, X^p + X_p\, {\eta_h}^p\, {}_{\Sigma_X}{\omega^k}_l{}^h{}_m\, {\eta^m}_q\, X^q
\end{aligned}$$

Section A.6 shows how to explicitly solve the diagonalized version of the aforementioned Riccati ODEs for an ample range of cases.

A.18 Proof of static portfolio optimization, Theorem 1

The goal is to maximize the expected utility U from terminal consumption

$$\sup_{\pi} U(\pi, W_t)$$

where expected utility U is taken by a CRRA investor over a consumption bundle

$$U(\pi, W_t) = E\left[u(v(W_T, \theta, P_T))\right],$$

the state process X_t determines risk free interest rate r_t, the risk premium $\mu_A - r_t\mathbb{1}$ and the drift of consumption prices μ_P

$$\begin{aligned} r_t =& \alpha_r + \beta_r^\intercal X_t \\ \mu_A - r_t\mathbb{1} =& \alpha_\Pi + \beta_\Pi X_t \\ \mu_P =& \alpha_P + \beta_P X_t \end{aligned}$$

subject to the following dynamics for consumption prices P_t, the state process X_t and the accumulated wealth W_t which depends on the investment strategy π

$$\begin{aligned} \frac{\mathrm{d}W_t}{W_t} &= \left(\pi^\intercal(\mu_A - r_t\mathbb{1}) + r_t\right)\mathrm{d}t + \pi^\intercal\Sigma_A\,\mathrm{d}Z_{A,t} \\ \mathrm{d}X_t &= \left(\alpha_X - \operatorname{diag}(\beta_X)X_t\right)\mathrm{d}t + \Sigma_X\,\mathrm{d}Z_{X,t} \\ \frac{\mathrm{d}P_t}{P_t} =& \mu_P\,\mathrm{d}t + \Sigma_P\,\mathrm{d}Z_{P,t}\,. \end{aligned}$$

Diffusion matrices, as well as correlations between the Wiener processes, are assumed to be constant.

Expected utility from implementing investment strategy π can be expanded

$$U(\pi, W_t) = E\left[\frac{\left(e^{(1-\theta^\intercal\mathbb{1})\log(1-\theta^\intercal\mathbb{1}) + \theta^\intercal\log(\theta) - \theta^\intercal\log(P_T)}W_T\right)^{1-\gamma}}{1-\gamma}\right]$$

and reformulated in terms of the integrated processes above

$$\begin{aligned} U(\pi, W_t) =& \frac{\left(e^{(1-\theta^\intercal\mathbb{1})\log(1-\theta^\intercal\mathbb{1}) + \theta^\intercal\log(\theta) - \theta^\intercal\log(P_t)}W_t\right)^{1-\gamma}}{1-\gamma} \\ &\cdot e^{(1-\gamma)\left(\pi^\intercal\alpha_\Pi + \alpha_r - \theta^\intercal\alpha_P - \frac{1}{2}\pi^\intercal\Sigma_A\Sigma_A^\intercal\pi + \frac{1}{2}\theta^\intercal\operatorname{diag}\left(\Sigma_P\Sigma_P^\intercal\right)\right)(T-t)} \\ &\cdot E\left[\exp\left(\begin{aligned} &(1-\gamma)\left(\pi^\intercal\beta_\Pi + \beta_r^\intercal - \theta^\intercal\beta_P\right)\int_t^T X_s\,\mathrm{d}s \\ &-(1-\gamma)\theta^\intercal\Sigma_P\int_t^T \mathrm{d}Z_{P,s} + (1-\gamma)\pi^\intercal\Sigma_A\int_t^T \mathrm{d}Z_{A,s} \end{aligned}\right)\right] \end{aligned}$$

Using Itô isometry we can see that

$$E\left[\int_t^T X_s\,\mathrm{d}s\right] = \frac{\alpha_X}{\beta_X}(T-t) + \operatorname{diag}\left(\frac{1-e^{-\beta_X(T-t)}}{\beta_X}\right)\left(X_t - \frac{\alpha_X}{\beta_X}\right) \tag{A.50}$$

$$
\operatorname{Var}\left[\int_t^T X_s\,\mathrm{d}s\right]_{i,j} = \begin{cases} \frac{\Sigma_{X,i,\cdot}\Sigma_{X,j,\cdot}^{\intercal}}{\beta_{X,i}\beta_{X,j}} \begin{pmatrix} T-t+\dfrac{1-e^{-(\beta_{X,i}+\beta_{X,j})(T-t)}}{\beta_{X,i}+\beta_{X,j}} \\ -\dfrac{1-e^{-\beta_{X,i}(T-t)}}{\beta_{X,i}}-\dfrac{1-e^{-\beta_{X,j}(T-t)}}{\beta_{X,j}} \end{pmatrix} & \text{if } \beta_{X,i}+\beta_{X,j}\neq 0 \\ \frac{\Sigma_{X,i,\cdot}\Sigma_{X,j,\cdot}^{\intercal}}{\beta_{X,i}\beta_{X,j}} \left(2(T-t)-\frac{1-e^{-\beta_{X,i}(T-t)}}{\beta_{X,i}}-\frac{1-e^{-\beta_{X,j}(T-t)}}{\beta_{X,j}}\right) & \text{if } \beta_{X,i}+\beta_{X,j}=0 \end{cases} \tag{A.51}
$$

$$
\operatorname{Cov}\left[\int_t^T X_s\,\mathrm{d}s\,,\int_t^T \mathrm{d}Z_{P,s}\right]_{i,j} = \frac{\Sigma_{X,i,\cdot}\rho_{XP,\cdot,j}}{\beta_{X,i}}\left(T-t-\frac{1-e^{-\beta_{X,i}(T-t)}}{\beta_{X,i}}\right) \tag{A.52}
$$

$$
\operatorname{Cov}\left[\int_t^T X_s\,\mathrm{d}s\,,\int_t^T \mathrm{d}Z_{A,s}\right]_{i,j} = \frac{\Sigma_{X,i,\cdot}\rho_{XA,\cdot,j}}{\beta_{X,i}}\left(T-t-\frac{1-e^{-\beta_{X,i}(T-t)}}{\beta_{X,i}}\right) \tag{A.53}
$$

and this makes it possible to resolve the expectation in the objective function as the expectation of a lognormal random variable. The logarithmic case from (2) can be solved in a similar way, except that the expectation is over a normal random variable. In both cases we arrive at

$$
U(\pi,W_t) = u\left(W_t \exp\begin{pmatrix} (1-\theta^{\intercal}\mathbb{1})\log(1-\theta^{\intercal}\mathbb{1})+\theta^{\intercal}\log(\theta)-\theta^{\intercal}\log(P_t) \\ +\left(\alpha_r-\theta^{\intercal}\alpha_P+\frac{1}{2}\theta^{\intercal}\operatorname{diag}\left(\Sigma_P\Sigma_P^{\intercal}\right)\right)(T-t) \\ +\left(\beta_r^{\intercal}-\theta^{\intercal}\beta_P\right)\left(\frac{\alpha_X}{\beta_X}(T-t)+\operatorname{diag}\left(\frac{1-e^{-\beta_X(T-t)}}{\beta_X}\right)\left(X_t-\frac{\alpha_X}{\beta_X}\right)\right) \\ +\frac{1}{2}(1-\gamma)\begin{pmatrix} \left(\beta_r^{\intercal}-\theta^{\intercal}\beta_P\right)\operatorname{Var}\left[\int_t^T X_s\,\mathrm{d}s\right]\left(\beta_r^{\intercal}-\theta^{\intercal}\beta_P\right)^{\intercal} \\ +\theta^{\intercal}\Sigma_P\Sigma_P^{\intercal}\theta(T-t) \\ -2\left(\beta_r^{\intercal}-\theta^{\intercal}\beta_P\right)\operatorname{Cov}\left[\int_t^T X_s\,\mathrm{d}s\,,\int_t^T \mathrm{d}Z_{P,s}\right]\Sigma_P^{\intercal}\theta \end{pmatrix} \\ +\pi^{\intercal}\psi_1-\dfrac{\pi^{\intercal}\psi_2\pi}{2} \end{pmatrix}\right) \tag{A.54}
$$

where

$$
\begin{aligned}
\psi_1 =& \alpha_\Pi(T-t)+\beta_\Pi\left(\frac{\alpha_X}{\beta_X}(T-t)+\operatorname{diag}\left(\frac{1-e^{-\beta_X(T-t)}}{\beta_X}\right)\left(X_t-\frac{\alpha_X}{\beta_X}\right)\right) \\
&+(\gamma-1)\begin{pmatrix} \left(\beta_\Pi\operatorname{Var}\left[\int_t^T X_s\,\mathrm{d}s\right]+\Sigma_A\operatorname{Cov}\left[\int_t^T X_s\,\mathrm{d}s\,,\int_t^T \mathrm{d}Z_{A,s}\right]^{\intercal}\right)\left(\beta_P^{\intercal}\theta-\beta_r\right) \\ +\left(\Sigma_A\rho_{PA}^{\intercal}\Sigma_P^{\intercal}(T-t)+\beta_\Pi\operatorname{Cov}\left[\int_t^T X_s\,\mathrm{d}s\,,\int_t^T \mathrm{d}Z_{P,s}\right]\Sigma_P^{\intercal}\right)\theta \end{pmatrix} \\
\psi_2 =& \gamma\Sigma_A\Sigma_A^{\intercal}(T-t)+(\gamma-1)\beta_\Pi\left(\operatorname{Var}\left[\int_t^T X_s\,\mathrm{d}s\right]\beta_\Pi^{\intercal}+2\operatorname{Cov}\left[\int_t^T X_s\,\mathrm{d}s\,,\int_t^T \mathrm{d}Z_{A,s}\right]\Sigma_A^{\intercal}\right)
\end{aligned}
$$

The first order conditions to maximize $U(\pi,W_t)$ imply that the optimal investment strategy is

$$
\pi^\star = \left(\frac{\psi_2^{\intercal}+\psi_2}{2}\right)^{-1}\psi_1
$$

Second order conditions show that $\pi^\star$ is indeed a maximum if the matrix ψ_2 is positive definite. A sufficient condition to make the matrix positive definite is $\beta_\Pi = 0$, since $\Sigma_A\Sigma_A^{\intercal}$ is assumed to be positive definite and $\gamma > 0$. Another sufficient condition is that $\gamma \geq 1$ and this is straightforward to notice when the matrix is rewritten as a sum of variances where the factor multiplying the second term is non-negative when $\gamma \geq 1$

$$\operatorname{Var}\left[\Sigma_A \int_t^T \mathrm{d}Z_{A,s}\right] + (\gamma - 1)\operatorname{Var}\left[\beta_\Pi \int_t^T X_s \,\mathrm{d}s + \Sigma_A \int_t^T \mathrm{d}Z_{A,s}\right].$$

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