Daily Leverage and Long-Term Investing using Leveraged Exchange Traded Funds

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Abstract

This paper explores the potential of leveraged Exchange Traded Funds (LETFs) for long-term investors and lifecycle portfolios. Leverage can increase welfare by enabling strategies that match the risk appetite of risk-tolerant investors, or by increasing financial wealth exposure to compensate for the illiquidity of human capital. We find LETFs to be suitable for both purposes with a caveat: risks associated to LETFs make it worthwhile typically only if the investor is sufficiently risk-tolerant. We also solve a dynamic portfolio optimization problem taking leverage costs and limits into account. We find that the optimal leverage target is fairly insensitive to typical leverage costs, and that welfare gains of relaxing leverage constraints are sizeable for risk tolerant investors. In our suitability analysis we study the risks of modelling discretely leveraged returns with geometric Brownian motion, as well as the probability of LETFs crashing over horizons of up to 40 years derived from extreme value theory and historical data.

1 Introduction

Leveraged instruments can increase welfare by enabling investment strategies that match the risk appetite of risk-tolerant investors, or by increasing financial wealth exposure to compensate for the illiquidity of human capital. On the downside, leverage magnifies losses and, depending on the implementation, investors can even end up with negative balances. The goal of this paper is to find an optimal policy that accounts for frictions in the form of leverage costs and analyze the risks involved.

We explore the use of leveraged exchange traded funds (LETFs) in order to achieve the benefits of leverage while avoiding potential pitfalls, in particular the risk of negative wealth. In a traditional discrete-time investment setting with stocks and bonds, any investor who wants to achieve more than 100% stock market exposure needs to finance this with a short position in bonds or borrowing on margin, thus risking to end up with a negative balance in case of adverse stock market returns. This is unfortunate since standard life-cycle models, based on continuous-time reasoning going back to e.g. Merton (1971), tend to recommend such overleveraged positions especially to young investors.

Compared to trading on margin, LETFs offer considerable advantages for retail investors including a partial solution to the overleverage dilemma. They relieve investors from the burdens of margin maintenance and avoid the possibility of negative wealth thanks to limited liability. Suppose an investor wants an exposure of 150% to the stock market. Instead of investing in stocks 150% and "shorting bonds" −50% or borrowing on margin an equivalent amount, investors can allocate 50% to a 1x ETF and 50% to a 2x LETF without the risk of wealth ever turning negative. Investors are also spared from the frequent trading necessary to maintain margins and their associated tax consequences, having more flexibility to decide when to realize capital gains.

Our portfolio optimization problem is based on Merton (1971). In the renowned Merton model, consumption and investment decisions can be solved separately and, under the mutual fund separation theorem in the absence of hedging motives, the investment decision is simply the fraction of wealth to invest in the risk-free asset and the tangent portfolio, which in equilibrium is the market portfolio (Merton, 1973). The optimal investment fraction in the market portfolio may involve leverage when investors are sufficiently risk tolerant or when the ratio of human capital to financial wealth is sufficiently high. Returns and investment strategies are modelled in continuous time, and those investment strategies need to be discretized when we apply them to the real world. In general, discretely leveraged daily returns are well approximated by continuously leveraged models. However whether Merton's framework is well suited to model daily leverage ultimately depends on how sensitive investors are to this approximation error, hence we also analyze the adequacy of this model for different types of investors. In line with conventional practice, we use the S&P 500 index as a proxy for the market portfolio because it is a sensible compromise between broad exposure to the global economy, liquidity and availability of financial instruments.

LETFs are financial products that aim to replicate a multiple of the underlying's daily returns, typically a factor 2 or 3, with daily compounding. As such, their returns can be well approximated by continuously leveraged portfolios in Merton's world. The accuracy of this model for compounded returns over long horizons has remained an open question, and these products have been typically relegated as instruments for short-term hedging or speculation only. Concerns are mainly about beta slippage, which relates to differences between discrete and continuously leveraged instruments, and volatility decay, which relates to differences between stochastic and deterministic processes. We believe that model errors are small in our case, since these errors depend on the distribution of the underlying returns and we only consider a relatively stable and well diversified underlying, which is a proxy for the market portfolio. To address previously mentioned concerns, we analyze the suitability of LETF for mean-variance and constant relative risk aversion investors over long-term horizons. This includes calculating the probability that a LETF crashes over different time horizons, as well as comparing the moments of the continuously compounded leveraged returns to their discrete analogues. We argue that beta slippage is mitigated by frequent compounding and that the apparent direct negative impact of volatility on leveraged returns is not something inherently wrong with LETFs; the same effect appears when leveraging in a Merton world and disappears when expected returns are computed.

Usage of LETFs expands the set of admissible strategies for non-margin investors but carries frictions comprising fees, tracking errors, as well as a leverage ceiling of typically 3 times the underlying. Accordingly we find that the optimal investment strategy is a downward corrected version of Merton's investment fraction. Frictions are inevitable because of imperfections in real-world markets and the limited liability of LETFs is equivalent to embedding an option component that must be paid one way or another. It is important to realize that LETFs are not pure derivative contracts on daily returns. Rather they are composite financial instruments that bundle existing derivative contracts in a limited liability product for the convenience of the investor, and do so efficiently. They are engineered to replicate leveraged daily returns, although they do so on a best effort basis without any explicit contractual guarantee. Yet it is quite natural to assume that most investors cannot borrow money, shortsell the underlying or manage margins more efficiently than LETFs do.

The decision to leverage along the capital market line can be justified on equilibrium capital asset pricing models (CAPM) such as the seminal work of Sharpe (1964) and Lintner (1965). Levy and Roll (2010) argue that "the market portfolio may be mean-variance efficient after all". Assuming that the market portfolio coincides with the tangent portfolio is not far fetched either. On the one hand, Black (1972) argues that under borrowing constraints, investors with an appetite for risk may depress the risk premium of high beta assets relative to those with low beta, placing the market portfolio beyond the tangent portfolio. But on the other hand, the existence of LETFs already shows that borrowing constraints are not so binding, and betting-against-beta does not seem to yield excess returns (Hou, Xue, & Zhang, 2020; Novy-Marx & Velikov, 2022). We assume that, despite leverage frictions, the optimal investment strategy lies on the capital market line. Tracking errors for the most extreme LETFs on assorted indices were on the order of 2% to 3% during the great recession (Avellaneda & Zhang, 2010). While this seems sizeable compared to the expected return of the underlying, it is a much smaller part of the leveraged expected returns, and a broad market index is arguably more stable than sector indexes. Moreover, finding a non-leveraged portfolio that can reliably maintain a high beta (e.g. 3) without introducing other types of non-rewarded risks seems a daunting task in practice.

Next we proceed with a brief literature review in Section 2. In Section 3 we describe the general model features and assumptions. In Section 4 we find the set of efficient portfolios with minimum costs at each leverage ratio and in Section 5 we solve a dynamic portfolio optimization problem. In Section 6 we evaluate the suitability of our LETF model for long term investors and find that LETFs can be appropriate for sufficiently risk-tolerant long term investors.

2 Literature review

Merton (1971) is one of the most influential papers in dynamic portfolio optimization, finding the optimal leverage and consumption strategies in closed form to maximize expected power utility under geometric Brownian motion (GBM). Portfolio optimization under borrowing and short-sales constraints was studied in Vila and Zariphopoulou (1997), Grossman and Vila (1992) and Teplá (2000). Other papers in this literature introduce inflation when only nominal bonds are available with stochastic interest rates (Brennan & Xia, 2002), or stochastic income (Henderson, 2005). Our setting has many connections to Cuoco and Liu (2000), who study portfolio optimization for investors trading on margin. Our contribution here consists in incorporating leverage costs as incurred by investors trading LETFs.

Usage of GBM to model LETF prices is already documented in Giese (2010), which they extend to capture borrowing and turnover costs. Avellaneda and Zhang (2010) derive discrete and continuous time approximations for LETF returns that allow more general stochastic processes for the price of the underlying, capturing fees and borrowing costs. They empirically validate their model for a wide variety of double and triple leveraged long and inverse LETF, finding mean absolute approximation errors of less than 4% in the years 2008 and 2009 that overlap with the a global financial crisis.

Regarding portfolio optimization with LETFs, Leung and Park (2017) find the optimal leverage ratio to maximize the long term-growth rate for a power utility investor under GBM, GARCH, CIR, Heston and other processes. Lundström and Peltomäki (2018) find the optimal leverage ratio to maximize the Kelly criterion or log-utility. Staden, Forsyth, and Li (2024) maximize the outperformance over standard investment benchmarks using LETFs. Guasoni and Mayerhofer (2023) offer a comprehensive overview of how LETFs work from a fund manager's perspective and derive a dynamic strategy to balance accumulated tracking differences against short-term tracking errors under proportional transaction costs. Dai, Kou, Soner, and Yang (2023) offer a dynamic strategy to minimize tracking errors taking into account quadratic transaction costs and nighttime market closure. Our additions to the portfolio optimization literature from the perspective of retail LETF holders comprise: posing the problem in a classical Merton framework with consumption, selection of LETFs according to an efficiency criterion and measuring welfare effects. Moreover, our complementary analyses help us to understand better how suitable the resulting investment policies are for different investors in terms of risk preferences.

There is an intense debate about how suitable LETFs are for long-term horizons, particularly for buy-and-hold investors. Concerns regarding beta slippage and volatility decay of LETFs have been studied by Avellaneda and Zhang (2010), Wagalath (2014), Guo and Leung (2014) and Salimian, Winder, Manakyan, and Khazeh (2019). Leung and Santoli (2012) use the VaR and CVaR within a GBM model to determine the admissible leverage ratio given a time horizon or vice versa. Loviscek, Tang, and Xu (2014) and Bansal and Marshall (2015) construct synthetic LETF returns based on historical data from the S&P 500 and other indices. Then they compare "naive" expected returns to compounded daily leveraged returns over different time horizons, where naive expected returns just multiply the underlying return over the full period by the leverage factor. They find that daily leveraged returns quickly diverge from naive expectations as the time horizon increases, and that they tend to overperform. While naive expectations are useful to understand how unsophisticated investors may perceive LETF, those naive investment strategies are slightly contrived. By not readjusting leverage ratios with respect to current portfolio wealth, naive strategies can use arbitrarily large daily leverage and turn portfolio wealth negative. We perform an analysis of LETF risks that is more appropriate for financially savvy investors by comparing how LETF returns differ from continuously leveraged portfolios, and studying the implications for investors with different types of risk preferences. We also address the concerns about beta slippage and, on top of it, we use extreme value theory to calculate the probability of LETF crashes over long horizons.

3 Setting

We explore the usage of LETFs for long term investors under short-selling constraints. Let us assume there are two reference financial indices: a constant risk-free interest rate r and a risky stock index S_t driven by a geometric Brownian motion process depending on the instantaneous expected market return μ_t and market volatility σ_t

$$\frac{\mathrm{d}S_t}{S_t} = \mu_t \,\mathrm{d}t + \sigma_t \,\mathrm{d}W_t \,.$$

Investors are expected utility maximizers with constant relative risk aversion (CRRA). They have a CRRA utility function with coefficient of relative risk aversion $\gamma > 0$. Informally, we refer to investors with a small but positive γ as risk tolerant.

$$u(x) = \begin{cases} \frac{x^{1-\gamma}}{1-\gamma} & \text{if } \gamma \neq 1 \\ \log(x) & \text{otherwise} \end{cases}. \tag{1}$$

There are also LETFs $k = 1, \dots, K$ with price $L_{k,t}$, leverage factor $\beta_k \in \mathbb{R}$ and leverage costs $f_k \in \mathbb{R}$ based on the risky stock index. The leverage factor of some instruments can be negative, which are known as inverse LETFs. We assume that investors cannot short-sell any financial instrument, or doing so is always less advantageous than maintaining nonnegative positions on inverse LETFs, both in terms of costs and capacity to reach higher leverage ratios than individually crafted leverage strategies. This is quite plausible if LETF managers achieve economies of scale and implement better risk management strategies. Without loss of generality we can treat investing in the risk-free asset and the stock index as special cases of LETFs with leverage factors 0 and 1 and possibly some leverage costs. That is, investors may not buy the stock index S_t directly, but instead they only have the possibility to buy a LETF with $\beta_k = 1$ and leverage costs f_k.

$$\mathrm{d}L_{k,t} = L_{k,t} \left((r_t + \beta_k(\mu_t - r_t) - f_k) \,\mathrm{d}t + \beta_k \sigma_t \,\mathrm{d}W_t \right) \tag{2}$$

The LETF price dynamics above are based on the continuous time models of Giese (2010) and Avellaneda and Zhang (2010). In Avellaneda and Zhang (2010) there are two sources of friction: fees and, for inverse ETFs, borrowing costs. Since LETF managers achieve negative exposures by means of derivatives such as swaps and futures instead of taking short positions, borrowing costs can be ignored in practice (Guasoni & Mayerhofer, 2023). Applying this model makes the price of a LETF under constant volatility coincide with a β_k leveraged portfolio in a Merton world minus some constant leverage costs f_k

$$\frac{L_{k,t}}{L_{k,0}} = \exp\left(\int_0^t r_s \,\mathrm{d}s + \beta_k \int_0^t \mu_s - r_s \,\mathrm{d}s - \beta_k^2 \int_0^t \frac{\sigma_s^2}{2} \,\mathrm{d}s - f_k t + \beta_k \int_0^t \sigma_s \,\mathrm{d}W_s \right). \tag{3}$$

This setting is very similar to the renowned portfolio problem of Merton (1971) when market parameters are constant, except for the short-selling constraints and availability of LETFs with their associated leverage costs. In that problem, consumption and investment decisions can be solved separately and, under the mutual fund separation theorem, the investment decision is simply the fraction of wealth to invest in the risk-free asset and the tangent portfolio, which is commonly assumed to be the market portfolio. When considering only financial wealth, the optimal investment fraction to invest in the market portfolio $\frac{\mu - r}{\gamma\sigma^2}$ is constant. Merton (1971) also solves the portfolio optimization problem for lifecycle investors endowed with riskless human capital H_t in addition to their initial financial wealth F_0. The optimal investment fraction with respect to their financial wealth F_t incorporates an additional multiplier

$$\frac{(\mu - r)}{\gamma\sigma^2} \frac{(F_t + H_t)}{F_t},$$

which, for typical young workers, implies leveraged positions since financial wealth F_t is relatively small in comparison to human capital H_t at the beginning of their working lives.

3.1 Data and parameters

We obtained historical S&P 500 returns from Center for Research in Security Prices (CRSP) and historical US LETF expense ratios from Refinitiv. We see that available leverage ratios vary from -3 to 3 and that expense ratios tend to be higher for more extreme leverage ratios. Historical overnight USD LIBOR data was obtained from MacroMicro.me and the source of SOFR data was the Federal Reserve Bank of New York. Following S&P Global guidelines (Spurrier, 2021), the risk free rate coincides initially with overnight LIBOR and becomes SOFR + 0.02963% from 2021-12-20 onwards. Realized variance is estimated from simple index returns during the previous 5 trading days[1]

$$\hat{\sigma}_t^2 = \frac{1}{5} \sum_{i=1}^{5} \left(R_{m,t-i\Delta t} \right)^2 .$$

Table 1 gives an overview of the US market for LETF instruments based on the S&P 500 index. It shows expense ratios from 2024 and historical difference statistics between 2019 and 2023 normalized by absolute leverage ratio, which makes them comparable in terms of risk exposure units. It is reasonable to expect that if there is a base error to replicate a certain index, then replicating a leveraged index will incur in an error that is at least a multiple of the base error. Historical difference statistics compare yearly aggregated differences between actual log returns and those implied by the continuous time model of Avellaneda and Zhang (2010). Positive (negative) differences imply that LETFs returns are higher (lower) than the theoretical continuous model. The columns *Discrete leverage* and *Tracking* break down those differences between discrete leverage effects and tracking differences. *Net tracking* accounts for tracking differences net of the expense ratio. Since expense ratios have been very stable over time, *Net tracking* is roughly equivalent to subtracting the latest expense ratios from *Tracking* differences.

[1]This differs slightly from Avellaneda and Zhang (2010). They use $\hat{\sigma}_t^2 = \frac{1}{5} \sum_{i=1}^{5} \left(R_{m,t-i\Delta t} \right)^2 - \bar{R}_{m,t}^2$ with $\bar{R}_{m,t} = \frac{1}{5} \sum_{i=1}^{5} R_{m,t-i\Delta t}$, but that measure could have a downwards bias. Either we assume that the expected return is zero and use the remaining degrees of freedom to estimate the variance, or we have to spend one degree of freedom to estimate the mean and the rest to estimate the variance. Given the scarcity of observations (only 5 business days) and the fact that drift $O(\Delta t)$ is negligible with respect to diffusion $O(\sqrt{\Delta t})$, the first choice is preferred.

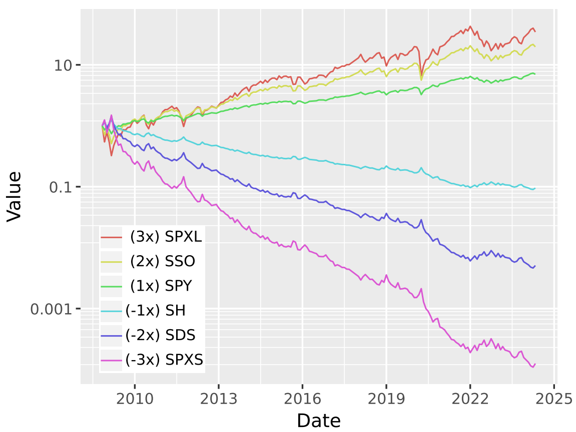
Table 1: LETFs normalized characteristics and statistics, US market

Ticker	Leverage	Norm. expense ratio	Normalized difference statistics			
			Model	Disc. leverage	Track.	Net Track.
UPRO	3	0.0031	-0.0043	0.0039	-0.0083	-0.0052
SPXL	3	0.0030	-0.0036	0.0039	-0.0075	-0.0042
SSO	2	0.0046	-0.0057	0.0023	-0.0080	-0.0035
SPUU	2	0.0030	-0.0037	0.0023	-0.0061	-0.0029
SPY	1	0.0009	-0.0005	0.0000	-0.0005	0.0004
IVV	1	0.0003	-0.0000	0.0000	-0.0000	0.0003
VOO	1	0.0003	-0.0001	0.0000	-0.0001	0.0002
SPLG	1	0.0002	0.0003	0.0000	0.0003	0.0006
SH	-1	0.0088	-0.0036	0.0058	-0.0094	-0.0005
SPDN	-1	0.0058	0.0002	0.0058	-0.0056	-0.0005
SDS	-2	0.0045	0.0037	0.0087	-0.0050	-0.0005
SPXS	-3	0.0036	0.0074	0.0114	-0.0040	-0.0004
SPXU	-3	0.0030	0.0078	0.0114	-0.0036	-0.0005

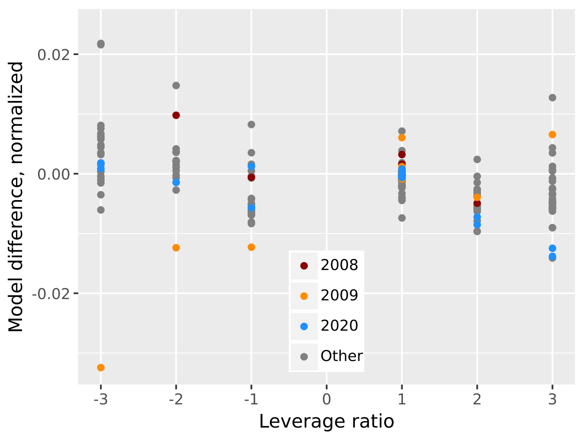
Expense ratio as of 2024-05-16. Statistics from 2020 to 2023.

Figure 1a shows the total return of LETFs with assorted leverage ratios over the last 15 years, and differences between actual and modelled log returns aggregated by product and year. During these 15 years, the an investment of \$1 in the 1x LETF has multiplied by 7.31, the 2x LETF has multiplied by 21.4 and the 3x LETF has multiplied by 38.99. Normalized model differences are distributed near the zero line, typically under 1%. Some years stand out: 2009 coincides with the end of a global financial crisis, and 2020 includes the stock market crash due to the COVID-19 pandemic. In general higher absolute leverage ratios are associated with positive model differences for negatively leveraged instruments and with negative model differences for positively leveraged instruments.

Figure 1: Historical performance and model differences



(a) Total return on \$1



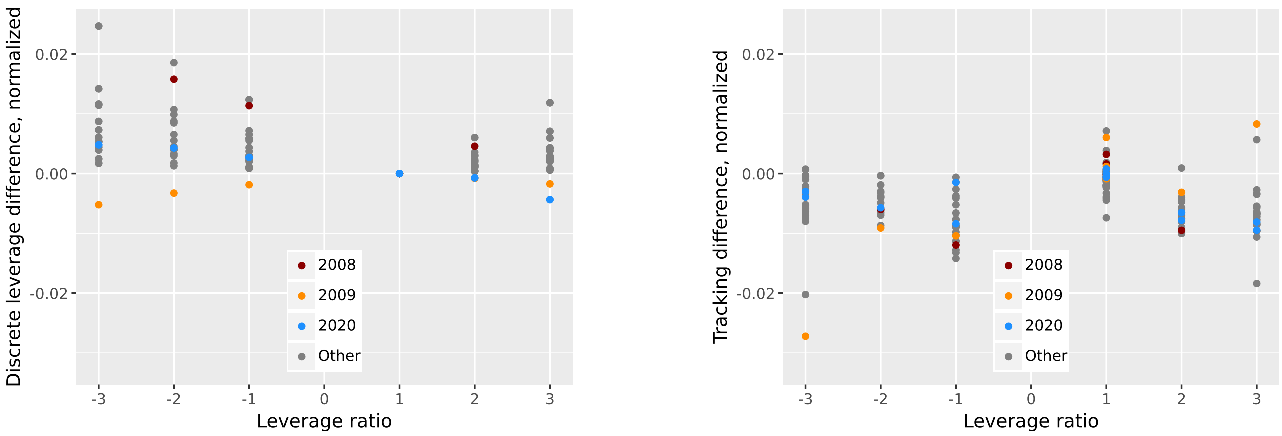
(b) Model differences

Figure 1a from 2008-11-05 to 2024-05-30. In Figure 1b each dot represents the cumulative differences for a LETF ticker and year from 2007 to 2023.

Model differences can be attributed to either discrete leverage effects or tracking differences.

Figure 2 shows that discrete leverage differences typically have a positive impact while tracking differences have usually a negative one. Negative discrete leverage differences coincide with periods of abrupt returns. The ending of the global financial crises had a negative impact across all instruments and the 2020 crash during the COVID-19 pandemic had a marked negative effect on 2x and 3x leveraged instruments. In terms of tracking differences, this pattern is not as clear. However it is noticeable that some instruments, particularly 1x ETFs, show occasionally positive tracking differences. Some possible explanations are that ETFs could be making extra profits by lending component shares to short sellers, or that some ETFs had a systematic bias towards some index components that were more profitable in this sample.

Figure 2: Model differences breakdown



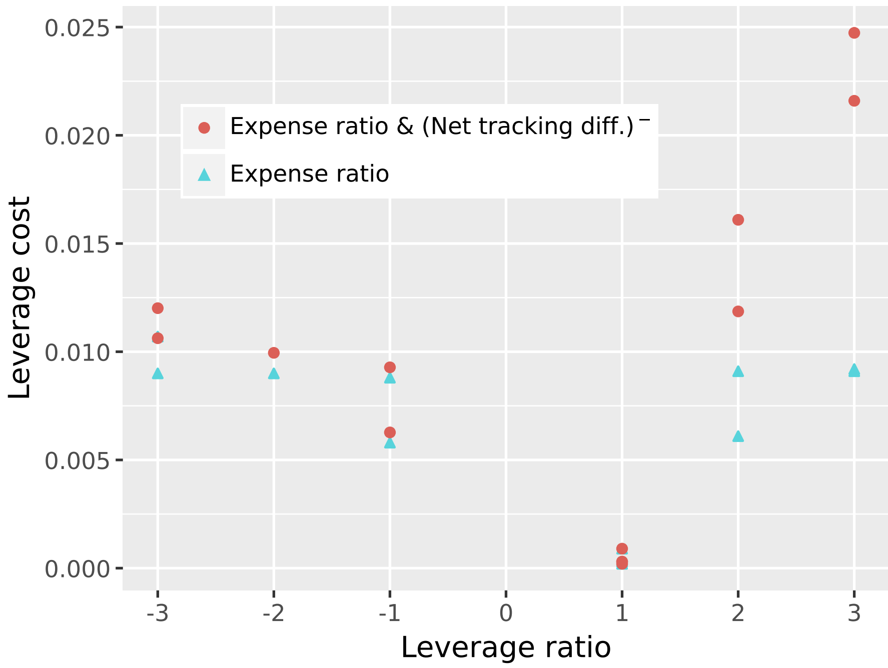
(a) Discrete leverage differences

(b) Tracking differences

We would like to summarize relevant costs in a single metric: expense ratios plus net tracking losses. Figure 3 compares this metric to plain expense ratios. The relevance of expense ratios and tracking differences is straightforward, however some investors could be concerned with noise in tracking difference estimates, particularly when these estimates are positive. A conservative way to address this point is to incorporate only the negative estimates, which we refer to as net tracking losses. We use data from 2020 to 2023 to estimate net tracking differences to balance relevance and accuracy. Extending the time window brings more observations making estimates more precise, but they may not incorporate improvements in replication technology and changes in tracking policies of different instruments (Dai et al., 2023; Guasoni & Mayerhofer, 2023).

There are good reasons not to take into account the apparent positive impact of discrete leverage. As we will see in Section 6.2, this effect is heavily influenced by extreme returns, which may not be adequately represented in this sample. We find that the risk of larger losses compensates these small but frequent gains.

Figure 3: Leverage costs, US market



Expense ratio as of 2024-05-16. Tracking difference statistics from 2019 to 2023.

In general we assume that investors can invest at the risk-free rate at zero cost, e.g. through TreasuryDirect[2]. For this reason, we add an extra instrument with leverage ratio $\beta_k = 0$ and zero cost $f_k = 0$. Additionally, illustrations and numerical examples assume constant parameters for the risk-free rate $r = 0.03$, instantaneous expected return $\mu = 0.1$ and volatility $\sigma = 0.2$ unless stated otherwise.

4 Efficient portfolios

An investor wants to find the portfolio with minimal leverage costs for each possible portfolio leverage ratio m subject to short-selling constraints. By assumption LETF instruments differ only in leverage ratio β and leverage costs f, while portfolios with identical leverage ratios and leverage costs are treated as equivalent, regardless of the underlying weights π. The investor has a universal distaste for portfolio leverage costs, such that, for any given target portfolio leverage ratio $m = \pi^\intercal \beta$ she strongly prefers the allocation π that minimizes portfolio costs $\pi^\intercal f$.

Despite its simplicity, this setting is broadly applicable to many portfolio optimization models as we explain in Remark 1 at the end of this section. It is also straightforward to apply in our model (3), since the dynamics of a portfolio $\pi \in \Delta^K$ depend only on the convex combination of leverage ratios $m = \pi^\intercal \beta$ and leverage costs $\pi^\intercal f$.

Finding the minimal leverage cost $f(m)$ for a given leverage ratio m, allows us to simplify the upcoming portfolio optimization problem in Section 5. Those problems can then be formulated

[2]This may not be applicable to all investors, particularly to those outside the U.S., however adjusting this fee to individual cases is straightforward.

in terms of a leverage ratio scalar m instead of a vector of portfolio weights π, e.g.

$$\max_{\pi \in \Delta^K} V(\pi^\intercal \beta, \pi^\intercal f) \qquad \rightarrow \qquad \max_{m \in [\beta_{\min}, \beta_{\max}]} V(m, f(m))$$

where $\beta_{\min}$ and $\beta_{\max}$ denote the lowest and highest leverage ratios among available instruments.

Back to the problem at hand, an investor wants to achieve a target portfolio leverage ratio m with minimum leverage costs subject to short-selling constraints. The market offers K instruments based on the selected underlying with leverage costs $f_1, \dots, f_K$ and leverage ratios $\beta_1, \dots, \beta_K$.

$$\begin{aligned} \min_{\pi \in \mathbb{R}^K} \; & \sum_{k=1}^{K} \pi_k f_k \\ \text{s.t.} \; & \sum_{k=1}^{K} \pi_k \beta_k = m \\ & \sum_{k=1}^{K} \pi_k = 1 \text{ and } \pi_1, \dots, \pi_K \geq 0 \end{aligned} \tag{4}$$

The decision variable is an investment strategy π that captures the investment fraction for each instrument. Short selling constraints $\pi_1, \dots, \pi_K \geq 0$ prevent arbitrage and make the optimal solution bounded. The risk-free asset and the non-leveraged stock index can be considered special instances of LETFs.

Definition 1 (Efficiency)**.**

- *Efficient LETF portfolios are convex combinations of LETF instruments such that the resulting portfolio leverage ratio cannot be replicated using a convex combination with strictly lower leverage costs.*
- *Efficient LETF instruments are those that constitute an efficient LETF portfolio by themselves alone.*
- *Locally efficient LETFs instruments are those with the lowest leverage cost among instruments with the same leverage ratio.*

Note that efficient LETF portfolios are composed exclusively by efficient LETF instruments. If that were not true, we could replace allocations to inefficient LETF instruments by efficient LETF portfolios of equal leverage ratio and reduce overall portfolio costs.

Lemma 1. *The set of efficient instruments is composed by*

i) the locally efficient instrument with the lowest leverage ratio β_{min}

ii) the locally efficient instrument with the highest leverage ratio β_{max}

iii) locally efficient instruments with leverage ratio β_k and leverage cost f_k such that, with respect to any two other locally efficient instruments l and h satisfying $\beta_l < \beta_k < \beta_h$, it holds that

$$f_k \leq \frac{\beta_h - \beta_k}{\beta_h - \beta_l} f_l + \frac{\beta_k - \beta_l}{\beta_h - \beta_l} f_h, \tag{5}$$

and leverage costs of efficient instruments are convex on leverage ratios.

Proof. Proof in Section A.1. □

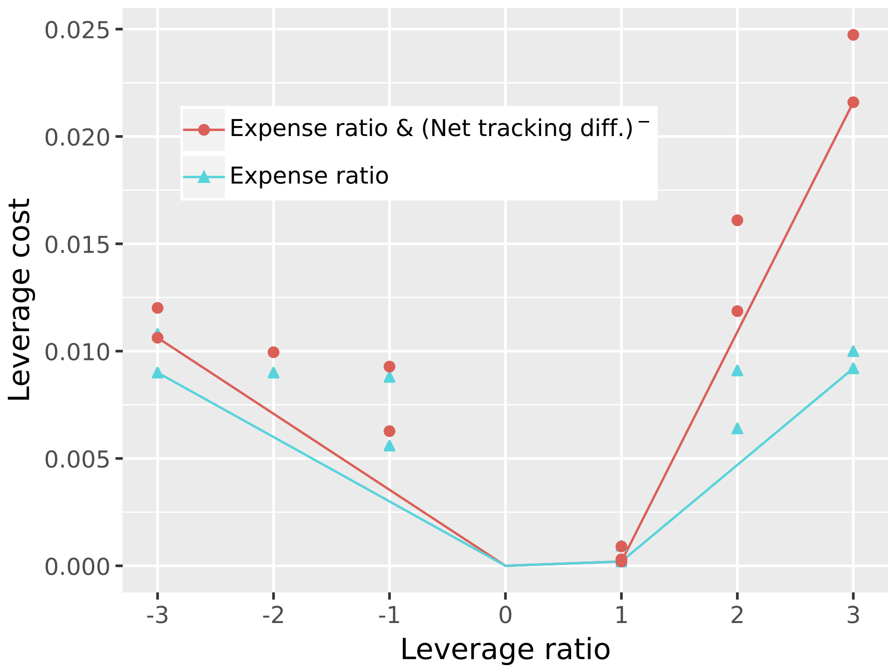
At this point, the solution to the simple linear problem in (4) is just one step away.

Theorem 1. *Minimum portfolio leverage costs as a function of portfolio leverage ratio are convex, and efficient LETF portfolios can be replicated with a convex combination of the two closest neighboring efficient LETFs. Let l and h denote the efficient instruments with leverage ratios just below and above the leverage target m respectively, the optimal portfolio is*

$$\pi_l^\star = \frac{\beta_h - m}{\beta_h - \beta_l} \qquad \pi_h^\star = \frac{m - \beta_1}{\beta_h - \beta_l}.$$

Proof. Proof in Section A.2. □

Figure 4: Efficient leverage costs, US market



Existing LETFs are represented as points. The lower convex hull is displayed as a solid line.

Efficient LETF portfolios lie in the lower contour of the convex hull and the minimum feasible cost f is convex in the leverage ratio β. This relationship is illustrated graphically in Figure 4. For the purposes of obtaining the efficient LETFs, the lower convex hull can be computed using Andrew's algorithm (Andrew, 1979) as described in Section A.3.

Corollary 1. *The optimal cost $f(m)$ as a function of the leverage ratio m is the piecewise linear function below. Given a list of K efficient LETFs in the sense of Definition 1 sorted by ascending leverage ratio,*

$$f(m) = \begin{cases} f_k & \textit{if } m = \beta_k \\ \frac{(\beta_{k+1}-m)f_k+(m-\beta_k)f_{k+1}}{\beta_{k+1}-\beta_k} & \textit{if } \beta_k < m < \beta_{k+1} \end{cases}. \tag{6}$$

Proof. Straightforward application of Theorem 1. □

For the remainder of the paper we restrict our attention to efficient LETFs using the notation laid out in Corollary 1. It is clear that we can ignore inefficient LETFs without loss of generality under our previous assumptions.

This result is broadly applicable to many portfolio optimization models, consider for instance the case below.

Remark 1. *Assume that the dynamics of the underlying risky asset and LETF's prices $L_{k,t}$ follow a stochastic process with constant leverage costs f_k*

$$\mathrm{d}L_{k,t} = L_{k,t}\left(\left(r_t + \beta_k(\mu_t - r_t) - f_k\right)\mathrm{d}t + \beta_k\sigma_t\,\mathrm{d}W_t\right),$$

and the desired exposure to the risky factor is m_t for the time horizon T. The dynamics of the portfolio depend on the investment strategy π_t subject to the constraint $\pi_t^\intercal\beta = m_t$

$$\mathrm{d}X_t = X_t\left(\left(r_t + m_t(\mu_t - r_t) - \pi_t^\intercal f\right)\mathrm{d}t + m_t\sigma_t\,\mathrm{d}W_t\right).$$

The investor can use Theorem 1 to maximize terminal wealth X_T by minimizing the cost difference $\pi_t^\intercal f$ with respect to an ideal frictionless market

$$X_T = X_0\exp\left(\int_0^T\left(r_t + m_t(\mu_t - r_t) - \frac{(m_t\sigma_t)^2}{2}\right)\mathrm{d}t - \int_0^T \pi_t^\intercal f\,\mathrm{d}t + \int_0^T m_t\sigma_t\,\mathrm{d}W_t\right). \quad (7)$$

The optimum in (7) *is achieved after replacing $\pi_t^\intercal f$ by $f(m_t)$ from* (6)*.*

5 Portfolio optimization

We consider the dynamic portfolio problem with consumption and/or terminal wealth of an expected utility maximizer in a market with continuous trading. There are no external inflows of wealth and the investment plan is self-financing. At every instant, the investor must choose a consumption rate $c_t \in [0,\infty)$ and leverage ratio $m \in [b_1, b_K]$. Leverage costs $f(m)$ can be in principle be described by any continuous piecewise once differentiable function with domain endpoints and interior breakpoints $b_1,\ldots,b_K$ in ascending order. This includes but it is not limited to the cost function defined in Corollary 1.

$$\frac{\mathrm{d}X_t}{X_t} = \left(r_t + m_t(\mu_t - r_t) - f(m_t) - c_t\right)\mathrm{d}t + m_t\sigma_t\,\mathrm{d}W_t \quad (8)$$

Market parameters $r_t \coloneqq r(Y_t), \mu_t \coloneqq \mu(Y_t,t), \sigma_t \coloneqq \sigma(Y_t,t)$ subject to $\sigma_t > 0$ are assumed to be finite and functions of a state vector process Y_t following a one-dimensional diffusion process

$$\mathrm{d}Y_t = z(Y_t)\,\mathrm{d}t + v(Y_t)\,\mathrm{d}W_t + \hat{v}(Y_t)\,\mathrm{d}\hat{W}_t.$$

Investor's wealth X_t follows the dynamics described in (8). The objective is to maximize expected utility from consumption and terminal wealth, weighted respectively by $\varepsilon_1, \varepsilon_2 \geq 0$ and subject to $\varepsilon_1 + \varepsilon_2 > 0$. Instantaneous and terminal utility are given by strictly continuous concave functions $u()$ and $u_T()$ with positive but diminishing marginal returns, discounted by impatience rate δ.

$$J(X_t, Y_t, t) = \sup_{m_t, c_t} E_t\left[\varepsilon_1\int_t^T e^{-\delta(s-t)}u\left(c_sX_s\right)\mathrm{d}s + \varepsilon_2 e^{-\delta(T-t)}u_T\left(X_T\right)\right] \quad (9)$$

Proposition 1. *Suppose that indirect utility (9) is finite, once differentiable with respect to time and twice differentiable with respect to wealth X_t and state Y_t, and also that the cost of leverage $f(m)$ is continuous weakly convex piecewise and once differentiable. The optimal implicit leverage ratio $m_t^\star$ and consumption rate $c_t^\star$ need to satisfy*

$$m_t^\star = \begin{cases} b_1 & \text{if } \mu_t - r_t \leq a_1^+. \\ \frac{\left(\mu_t - r_t - \frac{\partial f(m^\star)}{\partial m}\right)}{\sigma_t^2} \frac{\frac{\partial J(X_t,t)}{\partial X_t}}{\left(-\frac{\partial^2 J(X_t,t)}{\partial X_t^2}\right) X_t} - \frac{\frac{\partial^2 J(X_t,Y_t,t)}{\partial X_t \partial Y_t} v_t}{\frac{\partial^2 J(X_t,Y_t,t)}{\partial X_t^2} X_t \sigma_t} & \text{if } a_k^+ < \mu_t - r_t < a_{k+1}^- \\ b_k & \text{if } a_k^- \leq \mu_t - r_t \leq a_k^+ \\ b_K & \text{if } \mu_t - r_t \geq a_K^- \end{cases} \tag{10}$$

$$c_t^\star = (u_t')^{-1} \left(\varepsilon_1^{-1} \frac{\partial J(X_t,t)}{\partial X_t} \right) X_t^{-1}. \tag{11}$$

where ∂_- refers to the left partial derivative, ∂_+ refers to the right partial derivative, b_k denotes a breakpoint or domain endpoint of $f(m)$ and

$$a_k^\pm = \left. \frac{\partial_\pm f(m)}{\partial m} \right|_{m=b_k} - b_k \sigma_t^2 \frac{\frac{\partial^2 J(X_t,Y_t,t)}{\partial X_t^2} X_t}{\frac{\partial J(X_t,Y_t,t)}{\partial X_t}} - \sigma_t \frac{\frac{\partial^2 J(X_t,Y_t,t)}{\partial X_t \partial Y_t} v_t}{\frac{\partial J(X_t,Y_t,t)}{\partial X_t}}.$$

Indirect utility $J(X_t, Y_t, t)$ should satisfy the partial differential equation (PDE) below, where tr() *is the trace of a matrix,*

$$\begin{aligned} 0 =& \varepsilon_1 u\left(c_t^\star X_t\right) + \frac{\partial J(X_t,Y_t,t)}{\partial t} - \delta J(X_t,Y_t,t) \\ &+ \frac{\partial J(X_t,Y_t,t)}{\partial X_t} X_t \left(r_t + m_t^\star(\mu_t - r_t) - f(m_t^\star) - c_t^\star\right) + \frac{1}{2} \frac{\partial^2 J(X_t,Y_t,t)}{\partial X_t^2} X_t^2 m_t^{\star 2} \sigma_t^2 \\ &+ \frac{\partial J(X_t,Y_t,t)}{\partial Y_t} z_t + \frac{1}{2} \operatorname{tr}\left(\frac{\partial^2 J(X_t,Y_t,t)}{\partial Y_t^2} \left(v_t v_t^\intercal + \hat{v}_t \hat{v}_t^\intercal\right) \right) + \frac{\partial^2 J(X_t,Y_t,t)}{\partial X_t \partial Y_t} v_t X_t m_t^\star \sigma_t \end{aligned}$$

with boundary condition $J(X_T, Y_T, T) = \varepsilon_2 u_T(X_T)$.

Proof. See Section A.4. □

We can get some crude intuition about how the optimal policy in (10) changes depending on volatility conditions. Ignoring leverage frictions, overlooking changes to indirect utility $J(X_t, Y_t, t)$ and supposing that market price of risk and vol-vol v_t are unaltered, the interior of the candidate solution (10) suggests that investors should reduce their nominal stock exposure when volatility spikes to keep exposure to the underlying risk factor unchanged.

An explicit solution arises in Theorem 2 when considering investors with CRRA utility and constant market parameters r, μ, σ. The optimality of this classical solution can be established on the basis of a verification theorem, e.g. Pham (2009, Theorems 3.5.2 and 3.5.3).

Theorem 2. *For an expected CRRA utility maximizer individual under constant market parameters and a continuous weakly convex piecewise once differentiable cost of leverage $f(m)$,* *the optimal implicit leverage ratio $m_t^\star$ and explicit consumption rate $c_t^\star$ are*

$$m_t^\star = \begin{cases} b_1 & \text{if } m_M \le a_1^+ \\ m_M - \frac{1}{\gamma\sigma^2}\frac{\partial f(m^\star)}{\partial m} & \text{if } a_k^+ < m_M < a_{k+1}^- \\ b_k & \text{if } a_k^- \le m_M \le a_k^+ \\ b_K & \text{if } m_M \ge a_K^- \end{cases} \qquad \text{with} \qquad a_k^\pm = b_k + \frac{1}{\gamma\sigma^2}\frac{\partial_\pm f(m)}{\partial m}\bigg|_{m=b_k} \tag{12}$$

$$c_t^\star = \varepsilon_1^{\frac{1}{\gamma}} h(A_{m^\star}, t)^{-1},$$

where $m_M = \frac{\mu - r}{\gamma\sigma^2}$ is the Merton fraction, ∂_- refers to the left partial derivative, ∂_+ refers to the right partial derivative, b_k denotes a breakpoint or domain endpoint of $f(m)$, and

$$A_m = \frac{\delta - (1-\gamma)\rho_m}{\gamma} \tag{13}$$

$$\rho_m = r + m(\mu - r) - f(m) - \frac{1}{2}\gamma m^2\sigma^2 \tag{14}$$

$$h(A,t) = \begin{cases} \varepsilon_1^{\frac{1}{\gamma}} \frac{\left(1 - e^{-A(T-t)}\right)}{A} + \varepsilon_2^{\frac{1}{\gamma}} e^{-A(T-t)} & \text{if } A \neq 0 \\ \varepsilon_1^{\frac{1}{\gamma}}(T-t) + \varepsilon_2^{\frac{1}{\gamma}} & \text{otherwise} \end{cases}. \tag{15}$$

It yields an indirect utility of

$$J(X_t, t) = \begin{cases} \frac{\varepsilon_1}{A_{m^\star}^\gamma}\frac{X_t^{1-\gamma}}{1-\gamma} & \text{if } T \to \infty \text{ and } A > 0 \\ h(A_{m^\star}, t)^\gamma \frac{X_t^{1-\gamma}}{1-\gamma} & \text{if } T \text{ is finite} \end{cases}. \tag{16}$$

Proof. See Section A.5. □

Just as in Merton (1971), the optimal investment fraction (12) is constant, independent from the consumption policy and applies to both dynamic and static settings. Corollary 2 specializes Proposition 1 to the efficient cost function from Corollary 1, making the optimal investment fraction explicit in (17). Notice that leverage ratios β of efficient instruments are the breakpoints b of the leverage cost function.

Corollary 2. *For an expected CRRA utility maximizer individual under constant market parameters and the continuous, weakly convex and piecewise linear leverage cost function $f(m)$ defined in (6), the optimal explicit leverage ratio $m_t^\star$ and consumption rate $c_t^\star$ are*

$$m_t^\star = \begin{cases} \beta_1 & \text{if } m_M \le a_1^+ \\ m_M - \frac{1}{\gamma\sigma^2}\frac{f_{k+1} - f_k}{\beta_{k+1} - \beta_k} & \text{if } a_k^+ < m_M < a_{k+1}^- \\ \beta_k & \text{if } a_k^- \le m_M \le a_k^+ \\ \beta_K & \text{if } a_K^- \end{cases} \qquad \text{with} \qquad \begin{aligned} a_k^+ &= \beta_k + \frac{1}{\gamma\sigma^2}\frac{f_{k+1} - f_k}{\beta_{k+1} - \beta_k} \\ a_k^- &= \beta_k + \frac{1}{\gamma\sigma^2}\frac{f_k - f_{k-1}}{\beta_k - \beta_{k-1}} \end{aligned} \tag{17}$$

$$c_t^\star = \varepsilon_1^{\frac{1}{\gamma}} h(A_{m^\star}, t)^{-1},$$

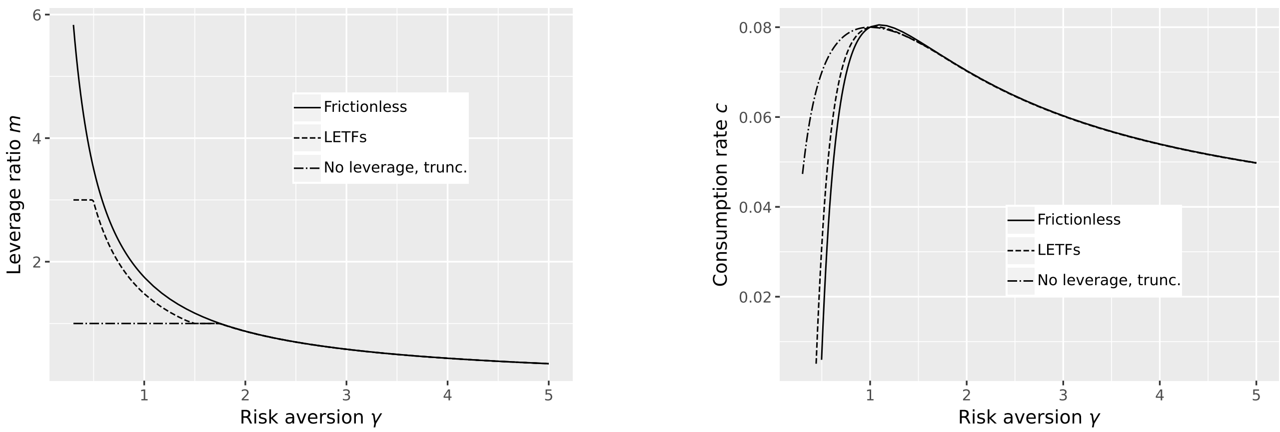
where $m_M = \frac{\mu - r}{\gamma\sigma^2}$ is the Merton fraction, β_k denotes a breakpoint or domain endpoint of $f(m)$, $h(A,t)$ is defined in (15),

$$A_m = \frac{\delta - (1-\gamma)\rho_m}{\gamma} \quad \text{and} \quad \rho_m = r + m(\mu - r) - f(m) - \frac{1}{2}\gamma m^2\sigma^2. \tag{18}$$

Proof. Straightforward specialization of Proposition 1 to piecewise linear cost function (6). □

Optimal leverage ratio and consumption rate from Corollary 2 are shown in Figure 5 for various coefficients of risk aversion on an infinite horizon setting. For risk averse investors, all models recommend a very similar policy as long as leverage is not necessary, and policy differences appear only as risk aversion decreases. The optimal leverage ratio sits between the frictionless Merton fraction and the truncated Merton fraction without leverage. Differences in leverage ratios are primarily due to binding leverage constraints corresponding to flat regions on the graph, while vertical shifts in curves correspond to the secondary impact of leverage costs. In terms of consumption rates, the optimal LETF investment plan appears to match quite close that of the frictionless Merton model, as seen in Figure 5b.

Figure 5: Solution comparison, infinite horizon



(a) Investment plan (b) Consumption rate

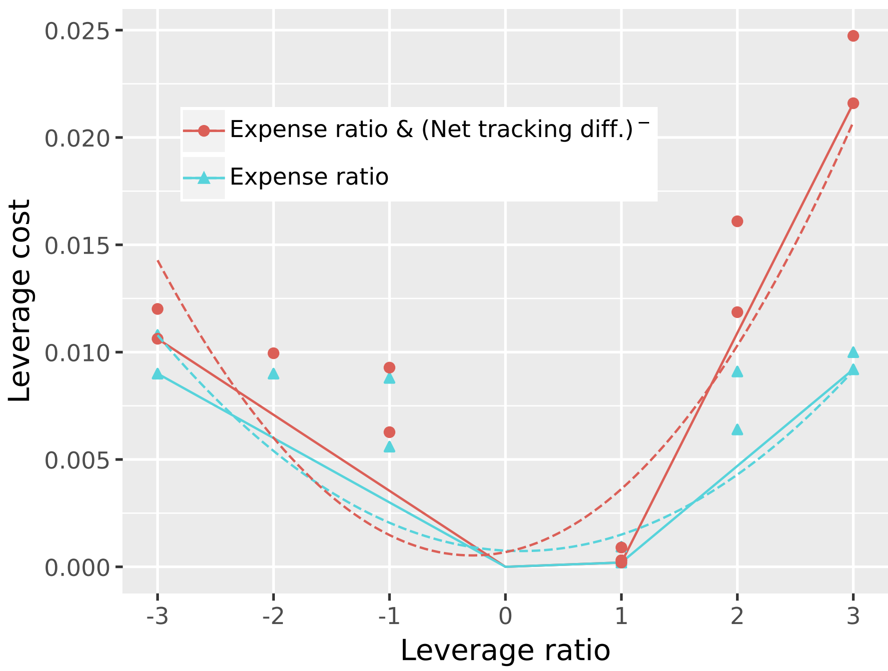
Using the optimal cost function $f(m)$ from Corollary 1.

We can also approximate leverage costs with a quadratic function

$$f(m) = k_0 + k_1 m + \frac{1}{2} k_2 m^2 \quad \text{where} \quad k_2 \geq 0. \tag{19}$$

Quadratic leverage provides a tractable approximation satisfying the convexity required by Lemma 1. The main purpose of (19) is to help us better understand its piecewise linear analogue (6), since the role of convexity is no longer obscured by breakpoints. Figure 6 shows a quadratic cost function fitted though ordinary least squares.

Figure 6: Quadratic approximation to leverage costs



Corollary 3. *For an expected CRRA utility maximizer individual under constant market parameters and quadratic leverage costs* (19)*, the optimal explicit leverage ratio* $m_t^\star$ *and consumption rate* $c_t^\star$ *are*

$$m_t^\star = \frac{\mu - r - k_1}{\gamma\sigma^2 + k_2} \qquad \textit{and} \qquad c_t^\star = \varepsilon_1^{\frac{1}{\gamma}} h(A_{m^\star}, t)^{-1},$$

where $h()$ *is defined in* (15)*,*

$$A_{m^\star} = \frac{\delta - (1-\gamma)\rho_{m^\star}}{\gamma} \quad \textit{and} \quad \rho_{m^\star} = r - k_0 + \frac{1}{2}\frac{(\mu - r - k_1)^2}{\gamma\sigma^2 + k_2}. \tag{20}$$

Proof. Straightforward specialization of Proposition 1 to quadratic cost function (19). □

The optimal $m_t^\star$ is a generalization of Merton's fraction where linear costs k_1 have an effect comparable to reducing the risk premium and quadratic costs k_2 an effect comparable to increasing the risk aversion.

5.1 Welfare implications

We evaluate welfare losses associated to leverage costs, welfare gains of lifting a maximum leverage constraint $\beta_{\max}$ and welfare gains from implementing sophisticated strategies that take the cost of leverage into account. This answers questions like what are the welfare gains of providing a 3x LETF technology to an investor who would otherwise be constrained to $m \in [0, 1]$, what is the marginal gain of further increasing this leverage threshold or if investors can neglect leverage costs when selecting an investment strategy.

This analysis takes place in a static setting with no consumption and no impatience rate. We define the certainty equivalent under investment strategy m by taking the inverse of the

CRRA utility (1) inverse $u^{-1}(\cdot)$ over the expected utility and discount it to present monetary units at the risk free rate. To simplify notation we assume that there are no frictions acting on the risk free rate, $f(0) = 0$, and we consider only constant investment fractions m. This is enough for this welfare analysis since optimal investment policies are constant in our setting and we only compare under optimality.

$$CE(m) = e^{-rT}\ u^{-1}\left(E[u(X_T(m))]\right) \tag{21}$$

Next, we use (16) to replace the expected utility in (21) under the investment policy m and leverage cost function f, arriving at the following expression

$$CE_f(m) = X_0 e^{\left(\rho_{m,f} - r\right)T} \quad \text{with} \quad \rho_{m,f} = r + m(\mu - r) - f(m) - \frac{1}{2}\gamma\sigma^2 m^2.$$

In this expression, $\rho_{m,f}$ can be interpreted as the "perceived" investment growth rate by the investor after adjusting for risk, and it extends our previous ρ_m by making the composition with the leverage cost function f explicit. Changes in welfare are measured as the logarithm of certainty equivalents, which can be interpreted as growth rates ρ when standardized by unit of time. The growth rate interpretation highlights the cumulative impact that leverage costs have.

$$\log\left(\frac{CE_{f_i}(m_i)}{CE_{f_j}(m_j)}\right)\frac{1}{T} = \rho_{m_i,f_i} - \rho_{m_j,f_j}$$

Welfare losses associated to leverage costs. These correspond to the drop in certainty equivalent wealth that investors suffer in comparison to a frictionless Merton world. In Merton's world, the optimal unrestricted investment strategy $m_M = \frac{\mu - r}{\gamma\sigma^2}$ yields a certainty equivalent of

$$CE_{\varnothing}(m_M) = X_0 e^{\left(\rho_{m_M,\varnothing} - r\right)T} \quad \text{with} \quad \rho_{m_M,\varnothing} = r + \frac{1}{2}\frac{(\mu - r)^2}{\gamma\sigma^2},$$

where the symbol $\varnothing$ indicates that there is no cost function in this setting.

Blindly following Merton's strategy in a world with leverage costs $f(m)$ incurs in welfare losses directly proportional to leverage costs

$$\rho_{m_M,f} - \rho_{m_M,\varnothing} = -f\left(m_M\right),$$

and sophisticated investors mitigate the impact of these leverage costs by adjusting down the leverage ratio to $m^\star$. By doing so, they introduce welfare losses due distortions in the investment policy

$$\rho_{m^\star,f} - \rho_{m_M,\varnothing} = -f(m^\star) - \frac{1}{2}\gamma\sigma^2\left(m_M - m^\star\right)^2. \tag{22}$$

Not surprisingly, carefully balancing leverage costs against policy distortions to minimize welfare losses, produces the same solution as the investment problem in (A.1). With quadratic leverage costs as in (19), welfare losses reduce to

$$\rho_{m^\star,f} - \rho_{m_M,\varnothing} = -\frac{\frac{1}{2}k_2\frac{(\mu-r)^2}{\gamma\sigma^2} + k_1(\mu - r - \frac{k_1}{2})}{\gamma\sigma^2 + k_2}.$$

As quadratic costs increase $k_2 \to \infty$ or linear costs grow within $k_1 \in [0, \mu - r]$, individuals are forced to choose $m = 0$. In this simplified setting, the marginal impact of linear and quadratic costs on welfare are $-m^\star$ and $-m^{\star 2}$ respectively.

Welfare gains from lifting the leverage constraint. Lifting the maximum leverage constraint from β_{ceil} to β'_{ceil} yields gains of $\rho_{\min(m^\star,\beta'_{\text{ceil}}),f} - \rho_{\min(m^\star,\beta_{\text{ceil}}),f}$. Figure 7a shows gains increasing monotonically in risk tolerance with the leverage cost function from Corollary 1. The marginal gains of increasing the leverage limit

$$\lim_{\beta'_{\text{ceil}}\downarrow\beta_{\text{ceil}}} \frac{\rho_{\min(m^\star,\beta'_{\text{ceil}}),f} - \rho_{\min(m^\star,\beta_{\text{ceil}}),f}}{\beta'_{\text{ceil}} - \beta_{\text{ceil}}} = \left(\mu - r - \gamma\sigma^2\beta_{\text{ceil}} - \left.\frac{\partial_+ f(m)}{\partial m}\right|_{m=\beta_{\text{ceil}}}\right) \mathbb{1}_{\beta_{\text{ceil}}<m^\star}$$

are non-negative due to the concavity of the investment problem in (A.1). Welfare gains of lifting the maximum leverage constraint are higher when marginal leverage costs are lower, when investors are more risk tolerant or when the leverage constraint was very restrictive.

Considering a quadratic cost function (19), welfare changes are

$$\rho_{\min(m^\star,\beta'_{\text{ceil}}),f} - \rho_{\min(m^\star,\beta_{\text{ceil}}),f} = \frac{1}{2}\left(\max(0, m^\star - \beta_{\text{ceil}})^2 - \max(0, m^\star - \beta'_{\text{ceil}})^2\right)(\gamma\sigma^2 + k_2)$$

and the marginal impact of loosening the maximum leverage constraint becomes

$$\lim_{\beta'_{\text{ceil}}\downarrow\beta_{\text{ceil}}} \frac{\rho_{\min(m^\star,\beta'_{\text{ceil}}),f} - \rho_{\min(m^\star,\beta_{\text{ceil}}),f}}{\beta'_{\text{ceil}} - \beta_{\text{ceil}}} = \max(0, \mu - r - k_1 - \beta_{\text{ceil}}(\gamma\sigma^2 + k_2)).$$

Welfare gains from sophistication. On top of subtracting wealth from investors, leverage costs can modify the investment strategy of rational investors. We refer to investors who take into account the cost of leverage when optimizing their portfolio as sophisticated, and as naive those who do not. These naive investors simply hold the a truncated Merton fraction in the stock index.

As illustrated in Figure 7b, sophistication gains $\rho_{m^\star,f} - \rho_{\min(m_M,\beta_{\max}),f}$ are typically small and concentrated on a narrow range of risk profiles. Very risk averse individuals want to invest little in the risk asset, and assuming that the risk free asset has no fees, there is simply little room to reduce any leverage cost at all. Highly risk tolerant individuals already find it optimal to use maximum leverage even when taking associated costs into account. If leverage costs were removed, these investors will continue to hit the same leverage constraint. Only those investors who are sufficiently risk tolerant to use leverage, but do so in moderation, benefit from sophistication.

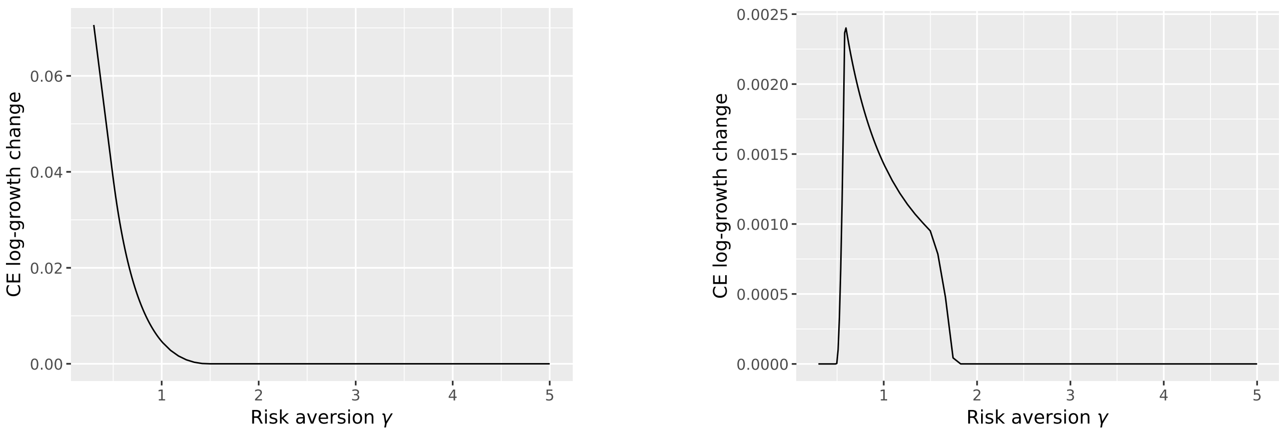
Considering a quadratic cost function (19), gains from sophistication simplify to

$$\rho_{m^\star,f} - \rho_{\min(m_M,\beta_{\max}),f} = \frac{1}{2}(\gamma\sigma^2 + k_2)(\min(m_M, \beta_{\max}) - m^\star)^2.$$

Since the sophisticated and naive Merton strategies are so similar, we can typically ignore the cost of leverage when deciding the optimal leverage ratio $m^\star$. This however does not mean that leverage costs are insignificant. The savings from the minimal cost strategy of Section 4 have a one-to-one impact on investor's welfare as shown in (22). The reason why they do not appear here, is that we already took the optimal $f(m)$ as given. This becomes more clear when we break down total welfare changes as

$$\rho_{m^\star,f} - \rho_{m_M,\varnothing} = \underbrace{\rho_{m^\star,f} - \rho_{\min(m_M,\beta_{\max}),f}}_{\text{Sophistication gains}}$$
$$+ \underbrace{\rho_{\min(m_M,\beta_{\max}),f} - \rho_{\min(m_M,\beta_{\max}),\varnothing}}_{\text{Direct naive costs, } -f(\min(m_M,\beta_{\max}))}$$
$$+ \underbrace{\rho_{\min(m_M,\beta_{\max}),\varnothing} - \rho_{m_M,\varnothing}}_{\text{Naive leverage constraint, } -\frac{1}{2}\gamma\sigma^2 \max(0, m_M - \beta_{\max})^2}$$

Figure 7: Welfare changes



(a) Gains lifting max. leverage $\beta_{\text{ceil}} = 1 \to 3$

(b) Gains from sophistication

Using the optimal cost function $f(m)$ from Corollary 1.

6 Suitability of LETFs for long term investment

When evaluating the appropriateness of LETFs for long term investors, we must take into account both the risks explicitly captured in the model as well as the risks that we have ignored or overlooked. The purpose of this section is to understand better these risks and find the type of investors for whom this model is appropriate, particularly among mean-variance and CRRA investors. More precisely, we analyze risks originating from modelling discrete returns using a continuous model. Even though we focus on LETFs, most of these results extend naturally to other investment strategies based on daily leverage. For instance, investors who trade on margin to overleverage as in Cuoco and Liu (2000) are exposed to discrete returns overnight and, contrary to LETFs, the lack of limited liability makes negative wealth a possibility in that setting.

Section 6.1 addresses the concerns about beta slippage and volatility decay. In Section 6.2 we evaluate how close is the distribution of discretely leveraged daily returns to its continuously leveraged analogue over long horizons. Our analysis covers risks inherent to discretely leveraging daily returns that are not captured by the continuous leverage model. The most salient risk is a LETF crash and we calculate the probability of such event over long horizons.

6.1 Beta slippage and volatility decay

Beta slippage relates to differences between discrete and continuously leveraged instruments, and it can be illustrated through a simple example. Suppose that the log-returns for a market index are r_1 and r_2 during two consecutive years. Additionally, suppose that the index went up during the first year, $r_1 > 0$ but reverted to the original level after the second year $r_2 \coloneqq -r_1$, that is,

$$r_1 + r_2 = 0. \tag{23}$$

Clearly, the wealth of an investor who held the market index during this period did not change and she obtained a return factor of $R = e^{r_1} e^{r_2} = 1$. But somewhat counterintuitively, the wealth of an investor discretely leveraging simple returns may be eroded.

If an investor had applied a leverage ratio $\beta > 1$ to simple returns, she would have obtained a return factor of

$$\begin{aligned} R =& \left(1 + \beta(e^{r_1} - 1)\right)\left(1 + \beta(e^{r_2} - 1)\right) \\ =& 1 + \beta(1 - \beta)\left((e^{r_1} - 1) + (e^{-r_1} - 1)\right). \end{aligned}$$

The compounded simple leveraged return depends on the path taken by returns, "buying high and selling low" along the way. Let us apply the well known inequality $e^x \geq 1 + x$, which is strict for $x \neq 0$. We can clearly see that paths with higher return deviations as well as higher leverage ratios result in higher losses

$$R = 1 + \underbrace{\beta(1 - \beta)}_{\leq 0}\underbrace{\left((e^{r_1} - 1) + (e^{-r_1} - 1)\right)}_{\geq 0} \leq 1.$$

If this investor had applied the leverage ratio $\beta > 1$ continuously in this deterministic setting, she would have matched the overall market return regardless of the precise leverage ratio or the actual path taken by the market index

$$R = \lim_{n\to\infty} \left(1 + \beta(e^{r_1 \frac{1}{n}} - 1)\right)^n \left(1 + \beta(e^{r_2 \frac{1}{n}} - 1)\right)^n = e^{\beta r_1} e^{\beta r_2} = 1$$

Continuous compounding in real-life may not be possible, but it approximates daily trading $n = 250$ quite well. Avellaneda and Zhang (2010) show theoretically and empirically that a continuous time approximation holds under some mild conditions in a more realistic setting with stochastic returns and without the simplistic restriction (23). They show that daily compounding mitigates the difference between discretely and continuously leveraged returns, and the approximation error is of the order of the standard deviation of daily returns, which is very small.

Applying Avellaneda and Zhang (2010) takes us directly to the concern of volatility decay, which relates to differences between stochastic and deterministic processes. Restricting our attention to GBM processes, this issue can be seen clearly in (3) when $L_{k,t}$ is reformulated in terms of the underlying price S_t.

$$\frac{L_{k,t}}{L_{k,0}} = \left(\frac{S_t}{S_0}\right)^{\beta_k} e^{-((\beta_k - 1)r + f_k)t} e^{-\beta_k(\beta_k - 1)\frac{\sigma^2}{2}t} \tag{24}$$

Compared to a deterministic world, the last term is driven by σ and indeed it has a negative impact on leveraged returns ($\beta \notin [0, 1]$), but this is not something inherently wrong with LETFs. Rather, it is simply a consequence of leveraging in a Merton world and the term originates as a byproduct of quadratic variation under Itô's lemma. On a expected return basis and ignoring leverage costs, the growth rate of LETF prices still scales linearly in the leverage ratio

$$E\left[\frac{L_{k,t}}{L_{k,0}}\right] = e^{(r+\beta_k(\mu-r)-f_k)t}.$$

An unfortunate limitation is that we cannot say much about volatility decay for general stochastic processes. We will revisit the beta slippage and volatility decay topics again in Section 6.2 under Lemma 2, when we compare the ratio of discrete and continuous return factors.

6.2 Discrete leverage error and LETF crashes

In this section we want to quantify the error incurred in approximating discretely leveraged daily returns, where the maximum operator captures limited liability,

$$R_d = \max\left(0, e^{r\Delta t} + \beta\left(e^{\left(\mu-\frac{\sigma^2}{2}\right)\Delta t+\sigma\sqrt{\Delta t}Z_t} - e^{r\Delta t}\right)\right) \tag{25}$$

using a continuous approximation (Avellaneda & Zhang, 2010; Giese, 2010) in a geometric Brownian motion world

$$R_c = e^{\left(r+\beta(\mu-r)-\beta^2\frac{\sigma^2}{2}\right)\Delta t+\beta\sigma\sqrt{\Delta t}Z_t},$$

as well as extending some parts of this analysis to historical returns wherever possible.

Table 2 shows that the mean and variance of discretely and continuously leveraged return factors compounded over selected time horizons are almost identical. For a mean-variance investor, the error of the continuous time approximation is almost negligible. Section A.6 contains the closed-form formulas needed for these computations.

Table 2: Measure ratios of discretely and continuously leveraged return factors

Horizon		Leverage ratio -3	-2	-1	1	2	3
1 years	$E[R_d]/E[R_c]$	0.9999	0.9999	1.0000	1.0000	1.0000	0.9999
	$SD[R_d]/SD[R_c]$	1.0008	1.0007	1.0006	1.0000	0.9995	0.9989
5 years	$E[R_d]/E[R_c]$	0.9994	0.9997	0.9999	1.0000	0.9999	0.9997
	$SD[R_d]/SD[R_c]$	1.0011	1.0008	1.0005	1.0000	0.9993	0.9978
20 years	$E[R_d]/E[R_c]$	0.9976	0.9988	0.9996	1.0000	0.9996	0.9988
	$SD[R_d]/SD[R_c]$	1.0034	1.0012	1.0004	1.0000	0.9983	0.9925
40 years	$E[R_d]/E[R_c]$	0.9953	0.9976	0.9992	1.0000	0.9992	0.9977
	$SD[R_d]/SD[R_c]$	1.0068	1.0023	1.0003	1.0000	0.9967	0.9851

For a market with 250 trading days per year and risky asset driven by geometric Brownian motion with risk-free rate $r = 0.03$, instantaneous expected return $\mu = 0.1$ and volatility $\sigma = 0.2$.

We also analyze the accuracy of the continuous model on a state-by-state basis. We do so through the discrete-continuous leverage ratio R_d/R_c, which tracks the direction and magnitude of discrete leverage deviations from the continuous time approximation.

Lemma 2. *The discrete-continuous leverage ratio* R_d/R_c *is quasi-concave (quasi-convex) on the discounted underlying return factor*

$$R_u = e^{\left(\mu - r - \frac{\sigma^2}{2}\right)\Delta t + \sigma\sqrt{\Delta t} Z_t}$$

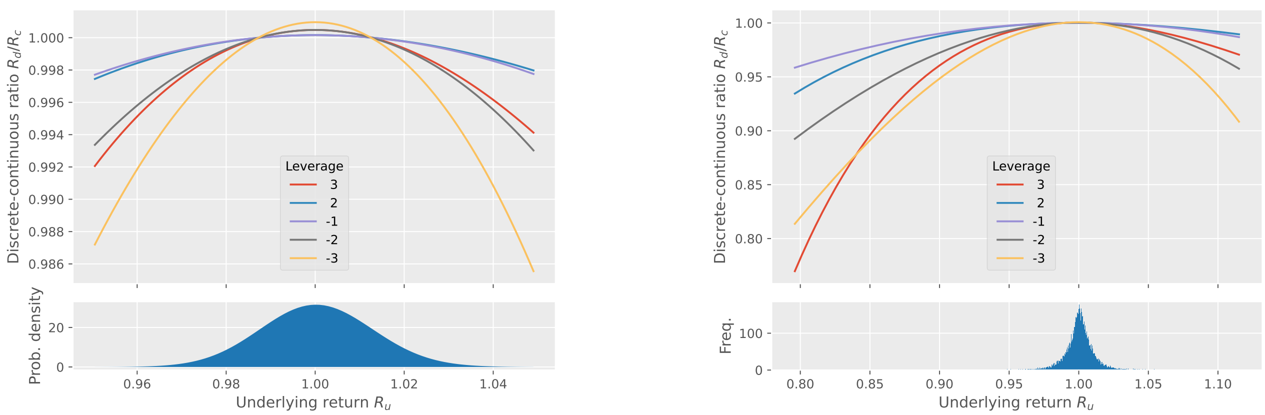
when $\beta \notin (0,1)$ *(*$\beta \in [0,1]$*), attaining a global maximum (minimum) when the discounted underlying return factor is* 1*. The relationship is strict when* $\beta \notin \{0,1\}$ *on the region not restricted by the max operator. Thus* R_d/R_c *is bounded above (below) by* $e^{(\beta^2-\beta)\frac{\sigma^2}{2}\Delta t}$ *when* $\beta \notin (0,1)$ *(*$\beta \in [0,1]$*). Additionally*

$$\lim_{R_u \uparrow \infty} \frac{R_d}{R_c} = \lim_{R_u \downarrow 0} \frac{R_d}{R_c} = \begin{cases} 0 & \text{if } \beta \notin [0,1] \\ \infty & \text{if } \beta \in (0,1) \\ 1 & \text{otherwise} \end{cases}$$

Proof. See Section A.7 □

Lemma 2 implies that, for conventional LETFs with $\beta \notin (0,1)$, discrete leverage errors are positive when the magnitude of daily returns is relatively small compared to volatility and negative otherwise. This is a manifestation of leverage slippage and volatility decay. Figure 8 illustrates how the discrete-continuous leverage ratio R_d/R_c depends on the underlying return[3]. The GBM probability density and the frequency of historical empirical returns give us an idea of how likely errors are. The negative skewness and high kurtosis of empirical returns are evident as the horizontal axis coincides with the range of observed historical returns. We can observe frequent instances of small but positive discrete leverage differences in agreement with Figure 2a. Next we study how this effect is expected to compound over long-term horizons.

Figure 8: Discrete-continuous ratio and underlying return



(a) Geometric Brownian motion

(b) Empirical

In the geometric Brownian motion setting, we consider a trading day in a market with 250 trading days per year. The probability density corresponds to that of a matching geometric Brownian motion process with risk-free rate $r = 0.03$, instantaneous expected return $\mu = 0.1$ and volatility $\sigma = 0.2$. The empirical panel is based on S&P 500 historical daily returns since 1962-07-03 until 2023-12-29, with an average annualized volatility $\hat{\sigma} = 0.1718$.

[3]By using the non-discounted underlying return we purposely overlook the effect of the risk-free rate. At short time intervals, the magnitude of drift $O(\Delta t)$ is negligible in comparison to Wiener increments $O(\sqrt{\Delta t})$.

Table 3 shows that on a state-by-state basis, discretely and continuously leveraged return factors compounded over selected time horizons tend to be similar. Whenever they are equivalent, the expectation of the R_d/R_c ratio should be close to 1 and the standard deviation close to 0. Computations for positive leverage ratios show that the expectation is indeed very close to 1 and the standard deviation is relatively small, with errors growing slowly in the time horizon. Since the expected CRRA utility converges to the expectation as risk aversion $\gamma \to 0$, we believe that the model precision should be acceptable for risk-tolerant expected CRRA utility maximizers. Section A.6 shows the formulas used for computing the moments of the R_d/R_c ratio.

Table 3: Discrete-continuous leverage ratio expectation and standard deviation

		Leverage ratio					
		-3	-2	-1	1	2	3
Horizon							
1 years	$\text{E}[R_d/R_c]$	0.9991	0.9997	1.0000	1.0000	1.0000	1.0000
	$\text{SD}[R_d/R_c]$	(0.0217)	(0.0108)	(0.0036)	(0.0000)	(0.0036)	(0.0108)
5 years	$\text{E}[R_d/R_c]$	0.9957	0.9986	0.9998	1.0000	1.0000	0.9998
	$\text{SD}[R_d/R_c]$	(0.0483)	(0.0241)	(0.0080)	(0.0000)	(0.0080)	(0.0240)
20 years	$\text{E}[R_d/R_c]$	0.9828	0.9946	0.9990	1.0000	1.0000	0.9994
	$\text{SD}[R_d/R_c]$	(0.0955)	(0.0481)	(0.0160)	(0.0000)	(0.0160)	(0.0481)
40 years	$\text{E}[R_d/R_c]$	0.9659	0.9892	0.9981	1.0000	1.0000	0.9988
	$\text{SD}[R_d/R_c]$	(0.1331)	(0.0676)	(0.0226)	(0.0000)	(0.0226)	(0.0680)

For a market with 250 trading days per year and risky asset driven by geometric Brownian motion with risk-free rate $r = 0.03$, instantaneous expected return $\mu = 0.1$ and volatility $\sigma = 0.2$.

After evaluating the general distribution of returns, we now focus on tail risks, particularly on the risk of a LETF crash. Even if the underlying index is always positive, we know from Lemma 2 that the price of a LETF can hit zero when the leverage ratio is outside the interval $[0, 1]$ and a daily return of the underlying is sufficiently extreme, e.g. -50% for 2x LETFs or -33% for 3x LETS. Once the LETF price hits zero, it will never recover and this is a source of concern for long term investors. Due to its absorbing nature, the most important question for us is whether investors will experience such an event at least once over their investment horizon, while the precise timing or count are less relevant.

The probability that a discretely leveraged instrument crashes or, more generally, yields ever a return factor R_d lower than threshold x over a given investment horizon T is tightly related to the extreme value distribution of the underlying index. In this setting, a threshold of $x = 0$ corresponds to a crash event. Let gains of the reference stock index during day t be $r^+_{S,t} := \log(S_t/S_{t-\Delta t})$, losses be $r^-_{S,t} := -r^+_{S,t}$, the highest daily gain and loss over a time horizon T be

$$M^{\pm}_{S,T} := \max_{t\in\{\Delta t,\ldots,T\}} \pm\, r_{S,t},$$

and the lowest daily discretely β leveraged return factor over a time horizon T be

$$M_{d,T} := \min_{t\in\{\Delta t,\ldots,T\}} R_{d,t},$$

with cumulative distribution functions $F_{r_S^\pm}$, $F_{M_{S,T}^\pm}$ and $F_{M_{d,T}}$. Using the definition of discretely leveraged return factors from (25) and the extreme value $M_{S,T}^\pm$, we obtain that the probability of a discretely leveraged instrument yielding ever a return factor $R_{d,t}$ below threshold x over time horizon T is

$$F_{M_{d,T}}(x) = \begin{cases} 1 - F_{M_{S,T}^-}\left(-\log\left(\frac{x+(\beta-1)e^{r\Delta t}}{\beta}\right)\right) & \text{if } \beta > \max(1 - xe^{-r\Delta t}, 0) \\ 0 & \text{if } \beta \in (0, 1 - xe^{-r\Delta t}] \\ \mathbb{1}_{x \geq e^{r\Delta t}} & \text{if } \beta = 0 \\ 1 & \text{if } \beta \in [1 - xe^{-r\Delta t}, 0) \\ 1 - F_{M_{S,T}^+}\left(\log\left(\frac{x+(\beta-1)e^{r\Delta t}}{\beta}\right)\right) & \text{if } \beta < \min(1 - xe^{-r\Delta t}, 0) \end{cases}$$

where cumulative distribution functions of extreme values, if daily returns are assumed to be independent, simplify to $F_{M_{S,T}^\pm}(x) = F_{r_S^\pm}(x)^{\frac{T}{\Delta t}}$.

Table 4 shows the probability of a LETF yielding ever a return factor below a given threshold or crashing (threshold 0) assuming a conventional geometric Brownian motion process. In this setting, the risk of a LETF crash is negligible and the explicit probability formula can be found in (A.8) under Section A.6.

Table 4: Discretely leveraged daily return factor, probability of falling at least once below threshold

		Leverage ratio					
		-3	-2	-1	1	2	3
Horizon	Thres.						
10 years	0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	0.5	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	0.8	0.0005	0.0000	0.0000	0.0000	0.0000	0.0001
	0.85	0.1404	0.0000	0.0000	0.0000	0.0000	0.0562
	0.87	0.6485	0.0009	0.0000	0.0000	0.0001	0.4159
40 years	0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	0.5	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	0.8	0.0018	0.0000	0.0000	0.0000	0.0000	0.0002
	0.85	0.4541	0.0001	0.0000	0.0000	0.0000	0.2065
	0.87	0.9847	0.0034	0.0000	0.0000	0.0005	0.8836

For a market with 250 trading days per year and risky asset driven by geometric Brownian motion with risk-free rate $r = 0.03$, instantaneous expected return $\mu = 0.1$ and volatility $\sigma = 0.2$.

Robust probability estimates of a LETF crash or return factor falling below a threshold, can be obtained applying Extreme Value Theory (EVT). A great reference in this domain is De Haan and Ferreira (2006) and we use it to analyze S&P 500 historical daily log-returns since 1962-07-03 until 2023-12-29. We estimate Pareto distributions to the tail of highest daily gains $r_{S,t}^+$ and losses $r_{S,t}^-$. It is well known that the empirical distribution of stock log-returns shows slightly fat tails and, using Hill's estimator, we obtain a shape parameter $\hat{\gamma}_- = 0.3342$ for the losses and $\hat{\gamma}_+ = 0.3127$ for the gains. Assuming that tails are regularly varying, the estimated Pareto distributions allow us to evaluate the upper tails of $F_{r_S^\pm}$ and the associated extreme quantile. The maximum value distribution over the $T/\Delta t$ time periods belongs to the Fréchet domain of attraction and it is approximated using De Haan and Ferreira (2006, Corollary 1.2.4) as

$$F_{M^{\pm}_{S,T}}(x) \approx \exp\left(- \left(\frac{x}{F^{-1}_{r^{\pm}_{S}}(1 - \Delta t/T)} \right)^{-1/\hat{\gamma}_{\pm}} \right).$$

Classical EVT tools were developed for independent and identically distributed (i.i.d.) observations, however later extensions are more flexible. Removing the independence assumption can even strengthen this analysis; for stationary sequences under mild mixing conditions, the univariate normalized maximum distribution $G^{\theta}(x)$ converges to that of its independent analogue $G(x)$ up to the extremal index $\theta \in (0, 1]$, which captures the degree of independence (Leadbetter, Lindgren, & Rootzén, 1983). Intuitively, dependence has an effect equivalent to reducing the effective sample size making the independent analogue an upper bound. For applications to log-returns from discrete stochastic volatility models see Andersen, Davis, Kreiß, and Mikosch (2009).

Using EVT in Table 5, we can see that over an horizon of 40 years the probability of 2x or 3x LETFs crashing is low but not negligible, while for -3x LETFs the risk is significant. In comparison to Table 4, risk increases across the board and the probability of encountering a return factor of 0.5 (50% loss) becomes sizeable.

Table 5: Extreme Value Theory probability of discretely leveraged daily return factor falling at least once below threshold

		Leverage ratio					
		-3	-2	-1	1	2	3
Horizon	Thres.						
10 years	0	0.0236	0.0085	0.0017	0.0000	0.0017	0.0085
	0.5	0.1430	0.0498	0.0085	0.0017	0.0237	0.0895
	0.8	0.8749	0.4772	0.0891	0.0499	0.3836	0.8210
	0.85	0.9917	0.7720	0.1866	0.1237	0.6964	0.9847
	0.87	0.9993	0.8929	0.2652	0.1891	0.8442	0.9985
40 years	0	0.0912	0.0337	0.0069	0.0000	0.0069	0.0337
	0.5	0.4605	0.1847	0.0337	0.0069	0.0913	0.3128
	0.8	0.9998	0.9253	0.3116	0.1851	0.8557	0.9990
	0.85	1.0000	0.9973	0.5622	0.4104	0.9915	1.0000
	0.87	1.0000	0.9999	0.7085	0.5677	0.9994	1.0000

For a market with 250 trading days per year and risk-free rate $r = 0.03$.

The possibility of a LETF crash does not necessarily rule out risk averse CRRA investors with $\gamma \geq 1$. At first sight, one could think the opposite since CRRA utility is defined exclusively for positive payoffs when $\gamma \geq 1$. If investors with $\gamma \geq 1$ can invest their entire wealth on a LETF, the risk of a LETF crash would be unfathomable and would make the continuous model inappropriate for them. Applying the continuous model to individuals with $\gamma \geq 1$ implicitly assumes that wealth dropping to zero is for all intents and purposes equivalent to an arbitrarily low but non-zero level of wealth. This argument invites us to reflect on why we use CRRA utility. CRRA utility is a tractable function that captures some stylized features commonly associated to human preferences such as diminishing marginal utility and constant relative risk aversion. However other properties may originate from mathematical artifacts unrelated to the real-world, such as the incommensurable difference between an empty wallet and having a cent when $\gamma \geq 1$. If investors can bear arbitrarily low levels of positive wealth but the possibility of zero wealth is utterly unacceptable, they can deposit one cent in a separate bank account and self-insure themselves for an almost negligible cost. To be clear, the analysis above still implies that there is a risk aversion threshold where the continuous LETF model ceases to be appropriate. The point is that this threshold does not have to be determined by some arbitrary mathematical artifacts related to CRRA utility.

To better illustrate this point, Table 6 compares numerically how well off are investors that use discretely leveraged instruments and safely put aside a small fraction, in comparison to those using continuously leveraged instruments. Consider an investor who is currently holding \$1 in a continuously leveraged instrument of a given leverage ratio. We calculate the premium π required to make the individual hold instead \$1 in a discretely leveraged instrument of the same leverage ratio. Including the premium, the expected utility of both portfolios over a single trading day is the same, but the discretely leveraged one is $1+\pi$ times more expensive. After using homogeneity to normalize by initial wealth, the ratio of discrete to continuous certainty equivalents becomes $CE_d/CE_c = (1+\pi)^{-\frac{1}{\Delta t}}$.

Table 6: Ratio of discrete to continuous certainty equivalents

	Leverage ratio					
	-3	-2	-1	1	2	3
Risk aversion						
0.10	0.9998	0.9999	1.0000	1.0000	1.0000	1.0000
0.50	0.9995	0.9998	1.0000	1.0000	1.0000	1.0000
1.00	0.9989	0.9997	0.9999	1.0000	1.0000	0.9999
5.00	0.9880	0.9970	0.9997	1.0000	0.9995	0.9963
10.00	0.9590	0.9903	0.9990	1.0000	0.9978	0.9841
20.00	0.8550	0.9653	0.9968	1.0000	0.9905	0.9353

For a market with 250 trading days where the risky asset is driven by geometric Brownian motion with risk-free rate $r = 0.03$, instantaneous expected return $\mu = 0.1$ and volatility $\sigma = 0.2$. Expectations were calculated numerically using Gauss-Kronrod adaptive quadrature and the premium was found using bisection.

We can also think about other ways to design a discrete time instrument that approximates R_c in a more reliable way and free from crash risk. Instead of leveraging simple daily returns, this alternative instrument could instead leverage log-returns as in the following equation

$$R_l = \exp\left(\beta \log\left(\frac{S_{\Delta t}}{S_0}\right) - (\beta - 1)\int_0^t r_s \,\mathrm{d}s - \beta(\beta-1)\int_0^t \frac{\sigma_s^2}{2}\,\mathrm{d}s\right).$$

The last term inside the exponential function makes sure that the instrument is arbitrage-free and coincides with (3) if leverage costs are excluded. Such an instrument would be much more appropriate for risk-averse investors as discretization crash risk is eliminated.

This thought exercise does not consider the difficulties and risks that fund managers could find when engineering such products. In comparison to a traditional LETF, log-leveraged instruments need an estimate of the daily realized variance $\sigma^2_{0,t}$, which on the one hand is less susceptible to manipulation than closing price but on the other hand it is not observable overnight. Replication using existing derivatives is also harder for log-leveraged ETFs. Daily payoffs of traditional LETF can closely match those of common futures or swaps after adjusting the exposure at the beginning of the day. But there are no widespread log-leveraged derivatives. Log-leveraged ETFs seem also harder to launch as derivative contracts on their own. Traditional LETFs enjoy the property that, ignoring limited liability, daily cashflows for a product with leverage ratio β are offset by those of its inversely leveraged twin $-\beta$ at equal investment values, that is, $R_{d,-\beta} - e^{r\Delta t} = -\left(R_{d,\beta} - e^{r\Delta t}\right)$. However this relationship does not hold for log-leveraged ETFs and it is unclear who would be interested in taking the other side of the contract. Participants on the other side would maintain a constant instantaneous leverage ratio, increasing (decreasing) the notional amount when the market moves against them (in their favor).

7 Discussion

Overall, the continuous time model seems to hold reasonably even if returns are discretely leveraged. Ideally, investors would prefer LETFs that apply the leverage multiplier to daily logarithmic returns instead of simple returns. The fact that those instruments are not available in financial markets, makes leverage worthwhile only to mean-variance investors or to risk-tolerant CRRA utility maximizers. This is not an impediment in practice, since typically risk-tolerant investors are the only ones interested in overleverage, while risk-averse investors would not be interested even if frictions were removed.

An intricate case is that of moderately risk-averse investors with overleverage motives stemming from their human capital illiquidity. On the one hand, overleverage risks are concentrated in the first few years of their careers when their financial wealth comprises a very small fraction of their total wealth. A LETF crash would not imply that their entire wealth dissapears: the crash only affects the fraction of financial wealth that was invested in LETFs, but not their human capital. They can still use their labor wages to sustain immediate consumption and build up financial wealth anew. On the other hand, the risk of unlikely but large beta slippage or even a LETF crash is a concern of sizable disutility for moderately risk averse investors that is not captured in the continuous model as shown in Table 6. These investors will find it optimal to moderate the use of leverage, although the precise adjustment would need to be determined numerically.

The decision to use LETFs as a long-term investment may be rational from an individual point of view, but consequences on a macroeconomic level are more complicated. Mankiw and Zeldes (1991) show that stockholders' consumption is more volatile and correlated to stock market movements than that of non-stockholders. LETFs make it easier for investors to reach higher leverage ratios with the caveat that extreme discretely leveraged returns carry some adverse non-linear effects in comparison to continuously leverage models. A sufficiently negative daily return (close to) triggering a LETF crash, could theoretically exacerbate systemic risk through an additional drop in consumption. On the positive side, LETFs also bring sizable welfare gains to risk tolerant investors as shown in Section 5.1. One could also argue, in light of the stock market participation puzzle (Campbell, 2006; Haliassos & Bertaut, 1995), that restricting access to LETFs may be counterproductive as households do not get enough exposure to stocks in the first place.

8 Conclusion

We argue that leveraged market exposure over long horizons can be easily achieved with LETFs and we include these instruments in a continuous time portfolio optimization model based on Merton (1971). This model shows that welfare gains from lifting the leverage constraint are sizable for risk-tolerant investors. The continuous specification seems to hold well when considering the challenges of leveraging in a discrete time world, and it seems suitable for long-term investors that are sufficient risk-tolerant. Yet this limitation is typically inconsequential: it is mostly risk-tolerant investors who are interested in overleverage. Lifecyle models could also incorporate LETFs to compensate for the illiquidity of human capital, although risk-averse investors would reduce their leverage ratios in comparison to continuous model predictions, as discrete leverage risks are more unpalable to them.

A Appendix

A.1 Proof of efficient LETFs set, Lemma 1

For the purposes of building a set, we can safely restrict our analysis to unique leverage ratio β_k and costs f_k combinations. Among instruments with the same leverage ratio it is obvious that the one with lowest leverage costs can replace the rest while reducing portfolio costs, so we only need to consider locally efficient instruments.

Showing i) and ii) is straightforward. Replicating their extreme leverage ratios is only possible using an instrument with the same leverage ratio, since assigning a positive weight to a less extreme instrument would also require assigning a positive weight to a more extreme instrument that does not exist.

For statement iii), we can visualize the problem geometrically in the $\mathbb{R}^2$ plane considering all possible combinations of locally efficient instruments $\{(\beta_1, f_1), \ldots, (\beta_K, f_K)\}$. Efficient LETF portfolios correspond to the lower contour of the convex hull, since for any leverage ratio m spanned by the convex hull, they represent the minimum achievable cost. These efficient LETF portfolios are the geometric edges of the lower contour of the convex hull and can be replicated as the combination of the two nearest left and right vertices along the leverage ratio axis. Efficient LETF instruments correspond to locally efficient instruments sitting in the lower contour of the convex hull and geometrically they are the vertices thereof. Thus each efficient LETF instrument k with (β_k, f_k) satisfies the equation below with respect to any replicating convex combination of instruments l and h where $\beta_l < \beta_k < \beta_h$

$$f_k \leq \frac{(\beta_h - \beta_k)f_l + (\beta_k - \beta_l)f_h}{\beta_h - \beta_l}.$$

In turn, this relationship implies that leverage costs of efficient instruments are convex on leverage ratios.

Alternatively, we can provide a more extensive explanation for statement iii). First we need to prove that efficient portfolios can be replicated with at most two instruments. By virtue of linearity, any portfolio π with a given leverage ratio m can be disaggregated into a convex combination of several at-most-two-instruments-portfolios each with average leverage ratio m. This decomposition can proceed iteratively as follows.

1) Create an at-most-two-instruments-portfolio with leverage ratio m using the instruments of portfolio π with the lowest l and highest h leverage ratios. The weights are non-negative and uniquely determined by $w_l = \frac{\beta_h - m}{\beta_h - \beta_l}$ and $w_h = \frac{m - \beta_l}{\beta_h - \beta_l}$.

2) Create a new remainder portfolio π' from π by setting $\pi'_l = 0$ and $\pi'_h = \pi_h - \pi_l \frac{w_h}{w_l}$ if $\frac{w_h}{w_l} \le \frac{\pi_h}{\pi_l}$, or $\pi'_l = \pi_h \frac{w_l}{w_h}$ and $\pi'_h = 0$ otherwise. Then renormalize weights.

3) Repeat step 1) for portfolio π' if composed by more than two instruments.

Efficient portfolios can be replicated with at most two instruments since the leverage cost of any portfolio is bounded below by the component with the lowest leverage cost among its at-most-two-instruments-portfolios, which constitutes itself a valid portfolio.

Now we need to derive the condition that characterizes efficient instruments. Let k be an instrument with leverage ratio β_k and cost f_k. To be efficient, it needs to have a cost lower or equal than any at-most-two-instruments-portfolio with portfolio leverage ratio equal to β_k. Previously we restricted our attention to instruments with the lowest cost among those with equal leverage ratios, therefore we only need to compare against portfolios composed of two instruments. If one instrument has a leverage ratio strictly lower than β_k, the other must have a strictly higher leverage ratio and vice-versa, otherwise it will not be able to match β_k and satisfy the simplex constraint. Let instruments l and h be such that $\beta_l < \beta_k < \beta_h$, then the leverage ratio of instrument k can only be replicated using the convex combination of weights $w_l = \frac{\beta_h - \beta_k}{\beta_h - \beta_l}$ and $w_h = \frac{\beta_k - \beta_l}{\beta_h - \beta_l}$. Portfolio costs are then given by the weighted sum of instrument leverage costs, and instrument k is efficient only if

$$f_k \le \frac{(\beta_h - \beta_k) f_l + (\beta_k - \beta_l) f_h}{\beta_h - \beta_l}.$$

At this point it has become apparent that leverage costs of efficient instruments are convex on the leverage ratio. Parametrizing any instrument (β_k, f_k) in terms of the convex combination $\lambda \in [0, 1]$ of instruments l and h such that $\beta_l < \beta_h$, it immediately follows that

$$f_k \le \lambda f_l + (1 - \lambda) f_h.$$

A.2 Proof of minimum cost portfolio, Theorem 1

Problem (4) can be visualized geometrically in the $\mathbb{R}^2$ plane as finding the lowest feasible point $(m, \bar{f})$ along the vertical line m. The feasible region is the intersection of the vertical line m with the convex hull spanned by all possible convex combinations of points $\{(\beta_1, f_1), \ldots, (\beta_K, f_K)\}$. That is, the feasible region is a segment on the vertical line m whose extremes touch the upper and lower contour of the convex hull, and the solution coincides with lower extreme. By Lemma 1 we know that vertices of the convex hull correspond to efficient LETF instruments and edges are the convex combination of two vertices, so the solution can be achieved with a combination of the two closest left and right efficient LETF instruments along the leverage ratio axis.

Alternatively, we can provide a more extensive explanation. Note that efficient LETF portfolios are composed exclusively by efficient LETF instruments. If that were not true, we could replace allocations to inefficient LETF instruments by efficient LETF portfolios of equal leverage ratio and reduce overall portfolio costs.

We can use Lemma 1 to show that a candidate solution π investing fractions π_1, π_2, π_3 in at least three efficient instruments such that (5) holds and $\beta_1 < \beta_2 < \beta_3$ where β_1, β_3 are extremes within the trio, can be replicated or improved removing a non-neighboring extreme. The convex combination of any three instruments belonging to π forms a subportfolio with a leverage ratio of $m_0 = \frac{\sum_{k=1}^{3} \pi_k \beta_k}{\sum_{k=1}^{3} \pi_k}$.

If $\beta_1 \leq m_0 \leq \beta_2$, the convex combination of instruments that achieves the lowest leverage costs does not need instrument 3.

$$\begin{aligned}
&\min_{w} f_1 w_1 + f_2 w_2 + f_3 w_3 \\
&\text{s.t.} \sum_{j=1}^{3} w_j \beta_j = m_0 \\
&\qquad \sum_{j=1}^{3} w_j = 1 \text{ and } w_1, w_2, w_3 \geq 0
\end{aligned}$$

substituting w_1 and w_2 using the constraints yields

$$\begin{aligned}
&\min_{w_3} f_1 + (f_2 - f_1)\frac{m_0 - \beta_1}{\beta_2 - \beta_1} + w_3 \left(\frac{f_3 - f_1}{\beta_3 - \beta_1} - \frac{f_2 - f_1}{\beta_2 - \beta_1} \right) (\beta_3 - \beta_1) \\
&\text{s.t.}\, 0 \leq w_3 \leq \frac{m_0 - \beta_1}{\beta_3 - \beta_1}
\end{aligned}$$

where $w_3 = 0$ is optimal since applying (5) to instrument 2 implies that

$$\frac{f_2 - f_1}{\beta_2 - \beta_1} \leq \frac{f_3 - f_1}{\beta_3 - \beta_1}.$$

If $\beta_2 \leq m_0 \leq \beta_3$, the convex combination of instruments that achieves the lowest leverage costs does not need instrument 1.

$$\begin{aligned}
&\min_{w} f_1 w_1 + f_2 w_2 + f_3 w_3 \\
&\text{s.t.} \sum_{j=1}^{3} w_j \beta_j = m_0 \\
&\qquad \sum_{j=1}^{3} w_j = 1 \text{ and } w_1, w_2, w_3 \geq 0
\end{aligned}$$

substituting w_2 and w_3 using the constraints yields

$$\min_w f_3 - \frac{\beta_3 - m_0}{\beta_3 - \beta_2}(f_3 - f_2) + w_1 \left(\frac{f_3 - f_2}{\beta_3 - \beta_2} - \frac{f_3 - f_1}{\beta_3 - \beta_1} \right) (\beta_3 - \beta_1)$$
$$\text{s.t. } 0 \leq w_1 \leq \frac{\beta_3 - m_0}{\beta_3 - \beta_1}$$

where $w_1 = 0$ is optimal since applying (5) to instrument 2 implies that

$$\frac{f_3 - f_1}{\beta_3 - \beta_1} \leq \frac{f_3 - f_2}{\beta_3 - \beta_2}.$$

Thus, we can use induction to remove one of the non-neighboring extreme instruments until the portfolio is composed only by the two closest neighboring instruments. Denote with l and h the efficient instruments just below and just above respectively, the convex combination that matches m is

$$w_1^\star = \frac{\beta_2 - m}{\beta_2 - \beta_1} \qquad w_2^\star = \frac{m - \beta_1}{\beta_2 - \beta_1}.$$

All together, this implies that the function $f(m)$ describing minimum leverage costs in terms of leverage ratio $m \in [\beta_{min}, \beta_{max}]$ is continuous and coincides with a linear interpolation between neighboring efficient instruments. To prove that this function is convex, it is sufficient to show that the slope of this linear interpolation is non-decreasing. Consider any three consecutive efficient instruments such that $\beta_1 < \beta_2 < \beta_3$, by Lemma 1 we immediately see that indeed the slope is non-decreasing

$$\frac{f_2 - f_1}{\beta_2 - \beta_1} \leq \frac{f_3 - f_2}{\beta_3 - \beta_2}.$$

A.3 Lower convex hull algorithm

Algorithm 1 Lower convex hull of LETFs using Andrew's algorithm

Create the list of unique points $P = [(\beta_1, f_1), \ldots, (\beta_K, f_K)]$ sorted in ascending leverage order, where each point matches a leverage ratio β_k to the lowest possible leverage cost f_k available for that leverage ratio.
Denote with S the list of provisional non-dominated points and add (β_1, f_1) to it
Use $(\beta_{\text{end}}, f_{\text{end}})$ to denote the last element of S and $(\beta_{\text{end}-1}, f_{\text{end}-1})$ for the second-to-last element if it exists.

```
for (β_k, f_k) in P_{2:K} do
    while size(S) ≥ 2 and (f_end − f_end−1)/(β_end − β_end−1) ≥ (f_k − f_end)/(β_k − β_end) do
        Pop (β_end, f_end) from S
    end while
    Push (β_k, f_k) to S
end for

Output: S
```

A.4 Proof of portfolio optimization, Proposition 1

Applying the Bellman principle of optimality can transform indirect utility $J()$ into a recursive function over a small time step Δt

$$J(X_t, Y_t, t) = \sup_{m_t, c_t} \varepsilon_1 e^{-\delta \Delta t} u\left(c_t X_t\right) \Delta t + e^{-\delta \Delta t} E_t\left[J(X_{t+\Delta t}, Y_{t+\Delta t}, t + \Delta t)\right].$$

The Hamilton-Jacobi-Bellman (HJB) equation is obtained multiplying both sides by $e^{\delta \Delta t}$, subtracting $J(X_t, Y_t, t)$, dividing by Δt and taking $\lim_{\Delta t \downarrow 0}$.

$$\begin{aligned}\delta J(X_t, Y_t, t) = \sup_{m_t, c_t} & \varepsilon_1 u\left(c_t X_t\right) + \frac{\partial J(X_t, Y_t, t)}{\partial t} + \frac{\partial J(X_t, Y_t, t)}{\partial X_t} X_t\left(r_t + m_t(\mu_t - r_t) - f(m_t) - c_t\right) \\ & + \frac{1}{2}\frac{\partial^2 J(X_t, Y_t, t)}{\partial X_t^2} X_t^2 m_t^2 \sigma_t^2 + \frac{\partial J(X_t, Y_t, t)}{\partial Y_t} z_t + \frac{1}{2}\operatorname{tr}\left(\frac{\partial^2 J(X_t, Y_t, t)}{\partial Y_t^2}\left(v_t v_t^\intercal + \hat{v}_t \hat{v}_t^\intercal\right)\right) \\ & + \frac{\partial^2 J(X_t, Y_t, t)}{\partial X_t \partial Y_t} v_t X_t m_t \sigma_t\end{aligned}$$

Notice that, in the HJB expression above, we used Itô's lemma to resolve

$$\begin{aligned}&\lim_{\Delta t \downarrow 0} \frac{E_t\left[J(X_{t+\Delta t}, Y_{t+\Delta t}, t + \Delta t)\right] - J(X_t, Y_t, t)}{\Delta t} \\ &= \frac{\partial J(X_t, Y_t, t)}{\partial t} + \frac{\partial J(X_t, Y_t, t)}{\partial X_t}\frac{\partial X_t}{\partial t} + \frac{1}{2}\frac{\partial^2 J(X_t, Y_t, t)}{\partial X_t^2}\left(\frac{\partial X_t}{\partial W_t}\right)^2 + \frac{\partial J(X_t, Y_t, t)}{\partial Y_t}\frac{\partial Y_t}{\partial t} \\ &\quad + \frac{1}{2}\operatorname{tr}\left(\frac{\partial^2 J(X_t, Y_t, t)}{\partial Y_t^2}\left(\frac{\partial Y_t}{\partial W_t}\frac{\partial Y_t}{\partial W_t}^\intercal + \frac{\partial Y_t}{\partial \hat{W}_t}\frac{\partial Y_t}{\partial \hat{W}_t}^\intercal\right)\right) + \frac{\partial^2 J(X_t, Y_t, t)}{\partial X_t \partial Y_t}\frac{\partial Y_t}{\partial W_t}\frac{\partial X_t}{\partial W_t}^\intercal.\end{aligned}$$

The consumption and investment problems can be solved separately. The investment problem is

$$\begin{aligned}\sup_{m_t} & \frac{\partial J(X_t, Y_t, t)}{\partial X_t} X_t\left(m_t(\mu_t - r_t) - f(m_t)\right) + \frac{1}{2}\frac{\partial^2 J(X_t, Y_t, t)}{\partial X_t^2} X_t^2 m_t^2 \sigma_t^2 \\ & + \frac{\partial^2 J(X_t, Y_t, t)}{\partial X_t \partial Y_t} v_t X_t m_t \sigma_t\end{aligned} \tag{A.1}$$

and second order conditions should be satisfied by concavity since $f(m_t)$ is assumed to be convex and $\frac{\partial^2 J(X_t, Y_t, t)}{\partial X_t^2}$ to be negative[4]. Assuming that $f(m)$ is piecewise differentiable, the solution is implicitly given by the first order condition (FOC) taking into account the possibility that the inflection point happens at a domain breakpoint. Let $\frac{\partial_- f(m)}{\partial m} = \lim_{h \uparrow 0} \frac{f(m+h) - f(m)}{h}$ denote the left and $\frac{\partial_+ f(m)}{\partial m} = \lim_{h \downarrow 0} \frac{f(m+h) - f(m)}{h}$ the right hand side limits of marginal leverage cost, all

[4]Ultimately this follows from diminishing marginal utility of $u()$.

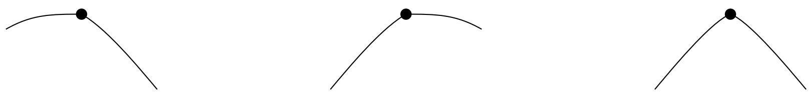
optimal points m must satisfy either

$$
\begin{aligned}
& 0 = \frac{\partial J(X_t, Y_t, t)}{\partial X_t} X_t \left(\mu_t - r_t - \frac{\partial_- f(m)}{\partial m} \right) + \frac{\partial^2 J(X_t, Y_t, t)}{\partial X_t^2} X_t^2 m_t \sigma_t^2 \\
& \quad + \frac{\partial^2 J(X_t, Y_t, t)}{\partial X_t \partial Y_t} \upsilon_t X_t \sigma_t
\end{aligned} \tag{A.2}
$$

$$
\begin{aligned}
\text{or} \quad & 0 = \frac{\partial J(X_t, Y_t, t)}{\partial X_t} X_t \left(\mu_t - r_t - \frac{\partial_+ f(m)}{\partial m} \right) + \frac{\partial^2 J(X_t, Y_t, t)}{\partial X_t^2} X_t^2 m_t \sigma_t^2 \\
& \quad + \frac{\partial^2 J(X_t, Y_t, t)}{\partial X_t \partial Y_t} \upsilon_t X_t \sigma_t
\end{aligned} \tag{A.3}
$$

$$
\text{or} \quad \frac{\partial_- f(m)}{\partial m} < \mu_t - r_t + \frac{m_t \sigma_t^2 \frac{\partial^2 J(X_t,Y_t,t)}{\partial X_t^2} X_t + \frac{\partial^2 J(X_t,Y_t,t)}{\partial X_t \partial Y_t} \upsilon_t \sigma_t}{\frac{\partial J(X_t,Y_t,t)}{\partial X_t}} < \frac{\partial_+ f(m)}{\partial m}. \tag{A.4}
$$

The critical points of the objective function correspond to: the left derivative being zero (A.2), the right derivative being zero (A.3), or the left derivative being positive while the right derivative is negative (A.4).



If both sided limits for $\frac{\partial f(m^\star)}{\partial m}$ coincide and m is within its differentiable region boundaries, we obtain an interior solution from (A.2) and (A.3)

$$
m_t^\star = \frac{\left(\mu_t - r_t - \frac{\partial f(m^\star)}{\partial m} \right)}{\sigma_t^2} \frac{\frac{\partial J(X_t,Y_t,t)}{\partial X_t}}{\left(-\frac{\partial^2 J(X_t,Y_t,t)}{\partial X_t^2} \right) X_t} - \frac{\frac{\partial^2 J(X_t,Y_t,t)}{\partial X_t \partial Y_t} \upsilon_t}{\frac{\partial^2 J(X_t,Y_t,t)}{\partial X_t^2} X_t \sigma_t}
$$

that, for the closest breakpoints or domain endpoints b_k below and b_{k+1} above relative to $m^\star$, satisfies the following inequalities

$$
a_k^+ < \mu_t - r_t < a_{k+1}^-
$$

where

$$
a_k^\pm = \left. \frac{\partial_\pm f(m)}{\partial m} \right|_{m=b_k} - b_k \sigma_t^2 \frac{\frac{\partial^2 J(X_t,Y_t,t)}{\partial X_t^2} X_t}{\frac{\partial J(X_t,Y_t,t)}{\partial X_t}} - \sigma_t \frac{\frac{\partial^2 J(X_t,Y_t,t)}{\partial X_t \partial Y_t} \upsilon_t}{\frac{\partial J(X_t,Y_t,t)}{\partial X_t}}.
$$

Then we evaluate the first condition at the endpoints of the domain, taking into account that only one sided limit is defined. From (A.3) and the second inequality of (A.4), the solution coincides with the lower endpoint $m^\star = b_1$ when

$$
\mu_t - r_t \le a_1^+.
$$

From (A.2) and the first inequality of (A.4), the solution coincides with the upper endpoint $m^\star = b_K$ when

$$
\mu_t - r_t \ge a_K^-.
$$

A breakpoint $m^\star = b_k$ of $f()$ in its interior domain is a solution when

$$a_k^- \leq \mu_t - r_t \leq a_k^+.$$

So the solution to the investment problem is

$$m_t^\star = \begin{cases} b_1 & \text{if } \mu_t - r_t \leq a_1^+. \\ \frac{\left(\mu_t - r_t - \frac{\partial f(m^\star)}{\partial m}\right)}{\sigma_t^2} \frac{\frac{\partial J(X_t,Y_t,t)}{\partial X_t}}{\left(-\frac{\partial^2 J(X_t,Y_t,t)}{\partial X_t^2}\right) X_t} - \frac{\frac{\partial^2 J(X_t,Y_t,t)}{\partial X_t \partial Y_t} v_t}{\frac{\partial^2 J(X_t,Y_t,t)}{\partial X_t^2} X_t \sigma_t} & \text{if } a_k^+ < \mu_t - r_t < a_{k+1}^- \\ b_k & \text{if } a_k^- \leq \mu_t - r_t \leq a_k^+ \\ b_K & \text{if } \mu_t - r_t \geq a_K^- \end{cases} \tag{A.5}$$

The consumption problem is

$$\sup_{c_t} \varepsilon_1 u\left(c_t X_t\right) - \frac{\partial J(X_t, Y_t, t)}{\partial X_t} X_t c_t$$

and second order conditions should be satisfied since the utility function is assumed to be strictly concave. The solution $c_t^\star$ depends on the inverse marginal utility function $(u')^{-1}()$

$$c_t^\star = (u_t')^{-1} \left(\varepsilon_1^{-1} \frac{\partial J(X_t, Y_t, t)}{\partial X_t} \right) X_t^{-1}. \tag{A.6}$$

Substituting the decision variables in the HJB with placeholders for the optimal consumption rate $c_t^\star$ from (A.6) and the implicit optimal leverage ratio $m_t^\star$ from (A.5) yields this partial differential equation (PDE)

$$\begin{aligned} 0 =& \varepsilon_1 u\left(c_t^\star X_t\right) + \frac{\partial J(X_t, Y_t, t)}{\partial t} - \delta J(X_t, Y_t, t) \\ &+ \frac{\partial J(X_t, Y_t, t)}{\partial X_t} X_t \left(r_t + m_t^\star(\mu_t - r_t) - f(m_t^\star) - c_t^\star\right) + \frac{1}{2} \frac{\partial^2 J(X_t, Y_t, t)}{\partial X_t^2} X_t^2 m_t^{\star 2} \sigma_t^2 \\ &+ \frac{\partial J(X_t, Y_t, t)}{\partial Y_t} z_t + \frac{1}{2} \operatorname{tr} \left(\frac{\partial^2 J(X_t, Y_t, t)}{\partial Y_t^2} \left(v_t v_t^\intercal + \hat{v}_t \hat{v}_t^\intercal\right) \right) + \frac{\partial^2 J(X_t, Y_t, t)}{\partial X_t \partial Y_t} v_t X_t m_t^\star \sigma_t \end{aligned}$$

with boundary condition $J(X_T, Y_T, T) = \varepsilon_2 u_T\left(X_T\right)$.

A.5 Proof of portfolio optimization with constant market params, Theorem 2

Assuming a CRRA utility function for $u()$ and $u_T()$ as in (1) and constant market parameters ($r_t \coloneqq r, \mu_t \coloneqq \mu, \sigma_t \coloneqq \sigma$), we continue from Section A.4. Both finite and infinite time horizon problems can be solved with the following ansatz, where $g(t)$ is an unspecified differentiable function of time,

$$J(X_t, t) = g(t)^\gamma \frac{X_t^{1-\gamma}}{1-\gamma}. \tag{A.7}$$

Second order conditions for the investment and consumption problems can be easily verified. The solutions to the investment and consumption problems are

$$
m^\star = \begin{cases} b_1 & \text{if } \frac{\mu-r}{\gamma\sigma^2} \le b_1 + \frac{1}{\gamma\sigma^2} \left.\frac{\partial_+ f(m)}{\partial m}\right|_{m=b_1} \\ \frac{\mu-r-\frac{\partial f(m^\star)}{\partial m}}{\gamma\sigma^2} & \text{if } b_k + \frac{1}{\gamma\sigma^2} \left.\frac{\partial_+ f(m)}{\partial m}\right|_{m=b_k} < \frac{\mu-r}{\gamma\sigma^2} < b_{k+1} + \frac{1}{\gamma\sigma^2} \left.\frac{\partial_- f(m)}{\partial m}\right|_{m=b_{k+1}} \\ b_k & \text{if } b_k + \frac{1}{\gamma\sigma^2} \left.\frac{\partial_- f(m)}{\partial m}\right|_{m=b_k} \le \frac{\mu-r}{\gamma\sigma^2} \le b_k + \frac{1}{\gamma\sigma^2} \left.\frac{\partial_+ f(m)}{\partial m}\right|_{m=b_k} \\ b_K & \text{if } \frac{\mu-r}{\gamma\sigma^2} \ge b_K + \frac{1}{\gamma\sigma^2} \left.\frac{\partial_- f(m)}{\partial m}\right|_{m=b_K} \end{cases}
$$

$$
c_t^\star = \varepsilon_1^{\frac{1}{\gamma}} g(t)^{-1}.
$$

Notice that $m^\star$ is constant across time and state X_t.

The PDE reduces to an ordinary differential equation (ODE) for $g(t)$

$$
0 = \varepsilon_1^{\frac{1}{\gamma}} + g'(t) - g(t)A \qquad \text{where} \qquad A = \frac{\delta - (1-\gamma)\left(r + m_t^\star(\mu - r) - f(m_t^\star) - \frac{1}{2}\gamma m_t^{\star 2}\sigma^2\right)}{\gamma}.
$$

In the infinite horizon version, the problem depends on the state but not on time. So $g'(t) = 0$, optimal consumption rate is

$$
c_t^\star = A \qquad \text{since} \qquad g(t) = \frac{\varepsilon_1^{\frac{1}{\gamma}}}{A}.
$$

The impact of utility from terminal wealth and distant indirect utility fades away as long as $A > 0$ since

$$
\lim_{t\to\infty} E\left[\varepsilon_2 e^{-\delta t} u(X_t)\right] = \lim_{t\to\infty} \varepsilon_2 e^{-At} \frac{X_0^{1-\gamma}}{1-\gamma} = 0
$$

and

$$
\lim_{t\to\infty} E\left[e^{-\delta t} J(X_t, t)\right] = \lim_{t\to\infty} e^{-At} \frac{\varepsilon_1}{A^\gamma} \frac{X_0^{1-\gamma}}{1-\gamma} = 0.
$$

The condition $A > 0$ also guarantees that the consumption rate is nonnegative $c^\star \ge 0$ and that indirect utility $J()$ is bounded. It implies an impatience rate satisfying

$$
\delta > (1-\gamma)\left(r + m_t^\star(\mu - r) - f(m_t^\star) - \frac{1}{2}\gamma m_t^{\star 2}\sigma^2\right).
$$

In the finite horizon version we use terminal utility at time T to derive the boundary condition

$$
g(T) = \varepsilon_2^{\frac{1}{\gamma}} \qquad \text{from} \qquad J(X_T, T) = g(T)^\gamma \frac{X_T^{1-\gamma}}{1-\gamma} = \varepsilon_2 \frac{X_T^{1-\gamma}}{1-\gamma},
$$

and then solve the ODE for $g(t)$ substituting the boundary $g(T)$ where necessary

$$
g(t) = \begin{cases} \varepsilon_1^{\frac{1}{\gamma}} \frac{1-e^{-A(T-t)}}{A} + \varepsilon_2^{\frac{1}{\gamma}} e^{-A(T-t)} & \text{if } A \neq 0 \\ \varepsilon_1^{\frac{1}{\gamma}} (T-t) + \varepsilon_2^{\frac{1}{\gamma}} & \text{otherwise} \end{cases}.
$$

It is easy to verify that $g(t) \geq 0$ for all A and thus $c^\star \geq 0$. Indirect utility J is bounded as long as problem parameters and A are bounded as well.

The function $g(t)$ can be generalized with respect to parameter A as $h(A,t)$ defined in (15). It is equivalent to the net present value with discount rate A of an annuity that pays ε_1 from time t to T and a final payoff ε_2 at time T.

Regardless of the time horizon, $m_t^\star$ and $c_t^\star$ are independent of wealth X_t and therefore wealth dynamics (8) follow a geometric Brownian motion, meaning that wealth always remains positive.

A.6 Discrete leverage error formulas

Here are the formulas used in Section 6.2 to evaluate the risk of discretely leveraged returns.

The moments of continuously leveraged return factors R_c come straight from the properties of the lognormal distribution.

$$E[R_c^n] = e^{n\left(r+\beta(\mu-r)+(n-1)\frac{\beta^2\sigma^2}{2}\right)T}$$

The moments of discretely leveraged return factors R_d are obtained in closed form as

$$
\begin{aligned}
E[R_d^n] &= E\left[\prod_{t=\Delta t,\dots,T}\left(\max\left(0,(1-\beta)e^{r\Delta t}+\beta e^{\left(\mu-\frac{\sigma^2}{2}\right)\Delta t+\sigma\sqrt{\Delta t}Z_t}\right)\right)^n\right] \\
&= \begin{cases}
\prod_{t=\Delta t,\dots,T}\int_{\frac{\log\left(\frac{(\beta-1)}{\beta}\right)}{\sigma\sqrt{\Delta t}}+\left(\frac{\sigma}{2}-\frac{\mu-r}{\sigma}\right)\sqrt{\Delta t}}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}\left(\beta e^{\left(\mu-\frac{\sigma^2}{2}\right)\Delta t+\sigma\sqrt{\Delta t}z}\right)^k\left((1-\beta)e^{r\Delta t}\right)^{n-k}\right)\phi(z)\,\mathrm{d}z & \text{if } \beta>1 \\
\prod_{t=\Delta t,\dots,T}\int_{-\infty}^{\frac{\log\left(\frac{(\beta-1)}{\beta}\right)}{\sigma\sqrt{\Delta t}}+\left(\frac{\sigma}{2}-\frac{\mu-r}{\sigma}\right)\sqrt{\Delta t}}\left(\sum_{k=0}^{n}\binom{n}{k}\left(\beta e^{\left(\mu-\frac{\sigma^2}{2}\right)\Delta t+\sigma\sqrt{\Delta t}z}\right)^k\left((1-\beta)e^{r\Delta t}\right)^{n-k}\right)\phi(z)\,\mathrm{d}z & \text{if } \beta<0 \\
\prod_{t=\Delta t,\dots,T}E\left[\sum_{k=0}^{n}\binom{n}{k}\left(\beta e^{\left(\mu-\frac{\sigma^2}{2}\right)\Delta t+\sigma\sqrt{\Delta t}Z_t}\right)^k\left((1-\beta)e^{r\Delta t}\right)^{n-k}\right] & \text{otherwise}
\end{cases} \\
&= \begin{cases}
\left(\sum_{k=0}^{n}\binom{n}{k}(1-\beta)^{n-k}e^{(n-k)r\Delta t}\beta^k e^{k\left(\mu+(k-1)\frac{\sigma^2}{2}\right)\Delta t}\int_{\frac{\log\left(\frac{(\beta-1)}{\beta}\right)}{\sigma\sqrt{\Delta t}}+\left(\frac{\sigma}{2}-\frac{\mu-r}{\sigma}\right)\sqrt{\Delta t}}^{\infty}\frac{e^{-\frac{(z-k\sigma\sqrt{\Delta t})^2}{2}}}{\sqrt{2\pi}}\,\mathrm{d}z\right)^{\frac{T}{\Delta t}} & \text{if } \beta>1 \\
\left(\sum_{k=0}^{n}\binom{n}{k}(1-\beta)^{n-k}e^{(n-k)r\Delta t}\beta^k e^{k\left(\mu+(k-1)\frac{\sigma^2}{2}\right)\Delta t}\int_{-\infty}^{\frac{\log\left(\frac{(\beta-1)}{\beta}\right)}{\sigma\sqrt{\Delta t}}+\left(\frac{\sigma}{2}-\frac{\mu-r}{\sigma}\right)\sqrt{\Delta t}}\frac{e^{-\frac{(z-k\sigma\sqrt{\Delta t})^2}{2}}}{\sqrt{2\pi}}\,\mathrm{d}z\right)^{\frac{T}{\Delta t}} & \text{if } \beta<0 \\
\left(\sum_{k=0}^{n}\binom{n}{k}(1-\beta)^{n-k}e^{(n-k)r\Delta t}\beta^k e^{k\left(\mu+(k-1)\frac{\sigma^2}{2}\right)\Delta t}\right)^{\frac{T}{\Delta t}} & \text{otherwise}
\end{cases} \\
&= \begin{cases}
\left(\sum_{k=0}^{n}\binom{n}{k}(1-\beta)^{n-k}e^{(n-k)r\Delta t}\beta^k e^{k\left(\mu+(k-1)\frac{\sigma^2}{2}\right)\Delta t}\Phi\left(-\frac{\log\left(\frac{(\beta-1)}{\beta}\right)}{\sigma\sqrt{\Delta t}}-\left(\frac{\sigma}{2}-\frac{\mu-r}{\sigma}\right)\sqrt{\Delta t}+k\sigma\sqrt{\Delta t}\right)\right)^{\frac{T}{\Delta t}} & \text{if } \beta>1 \\
\left(\sum_{k=0}^{n}\binom{n}{k}(1-\beta)^{n-k}e^{(n-k)r\Delta t}\beta^k e^{k\left(\mu+(k-1)\frac{\sigma^2}{2}\right)\Delta t}\Phi\left(\frac{\log\left(\frac{(\beta-1)}{\beta}\right)}{\sigma\sqrt{\Delta t}}+\left(\frac{\sigma}{2}-\frac{\mu-r}{\sigma}\right)\sqrt{\Delta t}-k\sigma\sqrt{\Delta t}\right)\right)^{\frac{T}{\Delta t}} & \text{if } \beta<0 \\
\left(\sum_{k=0}^{n}\binom{n}{k}(1-\beta)^{n-k}e^{(n-k)r\Delta t}\beta^k e^{k\left(\mu+(k-1)\frac{\sigma^2}{2}\right)\Delta t}\right)^{\frac{T}{\Delta t}} & \text{otherwise}
\end{cases}
\end{aligned}
$$

The moments of the discrete-continuous leverage ratio R_d/R_c are

$$
E\left[\left(\frac{R_d}{R_c}\right)^n\right] = E\left[\prod_{t=\Delta t,\dots,T}\left(\frac{\max\left(0,(1-\beta)e^{r\Delta t}+\beta e^{\left(\mu-\frac{\sigma^2}{2}\right)\Delta t+\sigma\sqrt{\Delta t}Z_t}\right)}{e^{\left(r+\beta(\mu-r)-\beta^2\frac{\sigma^2}{2}\right)\Delta t+\beta\sigma\sqrt{\Delta t}Z_t}}\right)^n\right]
$$

$$
= \begin{cases}
\left(\int_{\frac{\log\left(\frac{(\beta-1)}{\beta}\right)}{\sigma\sqrt{\Delta t}}+\left(\frac{\sigma}{2}-\frac{\mu-r}{\sigma}\right)\sqrt{\Delta t}}^{\infty}\left(\sum_{k=0}^n\binom{n}{k}\left((1-\beta)e^{-\left(\beta(\mu-r)-\beta^2\frac{\sigma^2}{2}\right)\Delta t-\beta\sigma\sqrt{\Delta t}z}\right)^{n-k}\left(\beta e^{-\left((\beta-1)(\mu-r)-(\beta^2-1)\frac{\sigma^2}{2}\right)\Delta t+(1-\beta)\sigma\sqrt{\Delta t}z}\right)^k\right)\phi(z)\,\mathrm{d}z\right)^{\frac{T}{\Delta t}} & \text{if } \beta>1\\
\left(\int_{-\infty}^{\frac{\log\left(\frac{(\beta-1)}{\beta}\right)}{\sigma\sqrt{\Delta t}}+\left(\frac{\sigma}{2}-\frac{\mu-r}{\sigma}\right)\sqrt{\Delta t}}\left(\sum_{k=0}^n\binom{n}{k}\left((1-\beta)e^{-\left(\beta(\mu-r)-\beta^2\frac{\sigma^2}{2}\right)\Delta t-\beta\sigma\sqrt{\Delta t}z}\right)^{n-k}\left(\beta e^{-\left((\beta-1)(\mu-r)-(\beta^2-1)\frac{\sigma^2}{2}\right)\Delta t+(1-\beta)\sigma\sqrt{\Delta t}z}\right)^k\right)\phi(z)\,\mathrm{d}z\right)^{\frac{T}{\Delta t}} & \text{if } \beta<0\\
\left(E\left[\sum_{k=0}^n\binom{n}{k}\left((1-\beta)e^{-\left(\beta(\mu-r)-\beta^2\frac{\sigma^2}{2}\right)\Delta t-\beta\sigma\sqrt{\Delta t}Z_t}\right)^{n-k}\left(\beta e^{-\left((\beta-1)(\mu-r)-(\beta^2-1)\frac{\sigma^2}{2}\right)\Delta t+(1-\beta)\sigma\sqrt{\Delta t}Z_t}\right)^k\right]\right)^{\frac{T}{\Delta t}} & \text{otherwise}
\end{cases}
$$

$$
= \begin{cases}
\left(\sum_{k=0}^n\binom{n}{k}(1-\beta)^{n-k}\beta^k e^{-\left((n\beta-k)(\mu-r)-(n\beta^2-k)\frac{\sigma^2}{2}\right)\Delta t}e^{\frac{(k-n\beta)^2\sigma^2\Delta t}{2}}\int_{\frac{\log\left(\frac{(\beta-1)}{\beta}\right)}{\sigma\sqrt{\Delta t}}+\left(\frac{\sigma}{2}-\frac{\mu-r}{\sigma}\right)\sqrt{\Delta t}}^{\infty}\frac{e^{-\frac{\left(z-(k-n\beta)\sigma\sqrt{\Delta t}\right)^2}{2}}}{\sqrt{2\pi}}\,\mathrm{d}z\right)^{\frac{T}{\Delta t}} & \text{if } \beta>1\\
\left(\sum_{k=0}^n\binom{n}{k}(1-\beta)^{n-k}\beta^k e^{-\left((n\beta-k)(\mu-r)-(n\beta^2-k)\frac{\sigma^2}{2}\right)\Delta t}e^{\frac{(k-n\beta)^2\sigma^2\Delta t}{2}}\int_{-\infty}^{\frac{\log\left(\frac{(\beta-1)}{\beta}\right)}{\sigma\sqrt{\Delta t}}+\left(\frac{\sigma}{2}-\frac{\mu-r}{\sigma}\right)\sqrt{\Delta t}}\frac{e^{-\frac{\left(z-(k-n\beta)\sigma\sqrt{\Delta t}\right)^2}{2}}}{\sqrt{2\pi}}\,\mathrm{d}z\right)^{\frac{T}{\Delta t}} & \text{if } \beta<0\\
\left(\sum_{k=0}^n\binom{n}{k}(1-\beta)^{n-k}\beta^k e^{-\left((n\beta-k)(\mu-r)-(n\beta^2-k)\frac{\sigma^2}{2}\right)\Delta t}E\left[e^{(k-n\beta)\sigma\sqrt{\Delta t}Z_t}\right]\right)^{\frac{T}{\Delta t}} & \text{otherwise}
\end{cases}
$$

$$
= \begin{cases}
\left(\sum_{k=0}^n\binom{n}{k}(1-\beta)^{n-k}\beta^k e^{-\left((n\beta-k)(\mu-r)-((k-n\beta)^2+n\beta^2-k)\frac{\sigma^2}{2}\right)\Delta t}\Phi\left(-\frac{\log\left(\frac{(\beta-1)}{\beta}\right)}{\sigma\sqrt{\Delta t}}-\left(\frac{\sigma}{2}-\frac{\mu-r}{\sigma}\right)\sqrt{\Delta t}+(k-n\beta)\sigma\sqrt{\Delta t}\right)\right)^{\frac{T}{\Delta t}} & \text{if } \beta>1\\
\left(\sum_{k=0}^n\binom{n}{k}(1-\beta)^{n-k}\beta^k e^{-\left((n\beta-k)(\mu-r)-((k-n\beta)^2+n\beta^2-k)\frac{\sigma^2}{2}\right)\Delta t}\Phi\left(\frac{\log\left(\frac{(\beta-1)}{\beta}\right)}{\sigma\sqrt{\Delta t}}+\left(\frac{\sigma}{2}-\frac{\mu-r}{\sigma}\right)\sqrt{\Delta t}-(k-n\beta)\sigma\sqrt{\Delta t}\right)\right)^{\frac{T}{\Delta t}} & \text{if } \beta<0\\
\left(\sum_{k=0}^n\binom{n}{k}(1-\beta)^{n-k}\beta^k e^{-\left((n\beta-k)(\mu-r)-((k-n\beta)^2+n\beta^2-k)\frac{\sigma^2}{2}\right)\Delta t}\right)^{\frac{T}{\Delta t}} & \text{otherwise}
\end{cases}
$$

Assuming a geometric Brownian motion, the cumulative distribution function $F_{M_{d,T}}$ of the discretely leveraged return factor R_d ever falling below threshold $x \geq 0$ at any time period $t \in \{\Delta t, \dots, T\}$ follows from the independence between return factors and lognormality.

$$
F_{M_{d,T}}(x) = \begin{cases}
1-\Phi\left(-\frac{\log\left(\frac{x+(\beta-1)e^{r\Delta t}}{\beta}\right)-\left(\mu-\frac{\sigma^2}{2}\right)\Delta t}{\sigma\sqrt{\Delta t}}\right)^{\frac{T}{\Delta t}} & \text{if } \beta>\max(1-xe^{-r\Delta t},0)\\
0 & \text{if } \beta\in(0,1-xe^{-r\Delta t}]\\
\mathbb{1}_{x\geq e^{r\Delta t}} & \text{if } \beta=0\\
1 & \text{if } \beta\in[1-xe^{-r\Delta t},0)\\
1-\Phi\left(\frac{\log\left(\frac{x+(\beta-1)e^{r\Delta t}}{\beta}\right)-\left(\mu-\frac{\sigma^2}{2}\right)\Delta t}{\sigma\sqrt{\Delta t}}\right)^{\frac{T}{\Delta t}} & \text{if } \beta<\min(1-xe^{-r\Delta t},0)
\end{cases} \tag{A.8}
$$

A.7 Proof of discrete-continuous leverage ratio bounds, Lemma 2

Let us express R_d/R_c using the discounted underlying return factor $R_u = e^{\left(\mu-r-\frac{\sigma^2}{2}\right)\Delta t+\sigma\sqrt{\Delta t}Z_t}$ as

$$
M = \frac{R_d}{R_c} = \frac{\max\left(0,1+\beta\left(R_u-1\right)\right)}{R_u^\beta}e^{(\beta^2-\beta)\frac{\sigma^2}{2}\Delta t}
$$

The quasi-convexity properties and extrema of $M = R_d/R_c$ as a continuous function of $R_u \in (0,\infty)$ can be established from the first derivative.

$$
M'(R_u) = \begin{cases}
-\frac{e^{(\beta^2-\beta)\frac{\sigma^2}{2}\Delta t}}{R_u^\beta}\beta(\beta-1)\frac{R_u-1}{R_u} & \text{if } 1+\beta\left(R_u-1\right)\geq 0\\
0 & \text{otherwise}
\end{cases}
$$

For $\beta \in \{0,1\}$, R_d/R_c is both weakly quasi-concave and quasi-convex since $M'(R_u) = 0$ everywhere. For $\beta \notin [0,1]$ and on the region not clipped by the max operator, the derivative $M'(R_u)|_{R_u<1} > 0$, $M'(R_u)|_{R_u=0} = 0$ and $M'(R_u)|_{R_u>1} < 0$, which makes M strictly quasi-concave and $R_u = 1$ a global maximum. For $\beta \in (0,1)$ and on the region not clipped by the max operator, the derivative $M'(R_u)|_{R_u<1} < 0$, $M'(R_u)|_{R_u=0} = 0$ and $M'(R_u)|_{R_u>1} > 0$, which makes M strictly quasi-convex and $R_u = 1$ a global minimum. Extending the domain to the region clipped by the max operator, $1 + \beta\,(R_u - 1) < 0$, which surrounds the core domain and satisfies $M'(R_u) = 0$, just weakens the quasi-concavity and quasi-convexity relationships. The upper and lower bounds values are obtained evaluating $M(R_u)$ at the global extremum $R_u = 1$.

The limits of $\frac{R_d}{R_c}$ at the extreme values of R_u can be easily verified after resolving the maximum operator

$$\lim_{R_u\uparrow\infty} \frac{R_d}{R_c} = \begin{cases} \lim_{R_u\uparrow\infty} \frac{\beta R_u}{R_u^\beta} e^{(\beta^2-\beta)\frac{\sigma^2}{2}\Delta t} = 0 & \text{if } \beta > 1 \\ \lim_{R_u\uparrow\infty} \frac{0}{R_u^\beta} e^{(\beta^2-\beta)\frac{\sigma^2}{2}\Delta t} = 0 & \text{if } \beta < 0 \\ \lim_{R_u\uparrow\infty} \frac{\beta R_u}{R_u^\beta} e^{(\beta^2-\beta)\frac{\sigma^2}{2}\Delta t} = \infty & \text{if } \beta \in (0,1) \\ 1 & \text{otherwise} \end{cases}$$

$$\lim_{R_u\downarrow 0} \frac{R_d}{R_c} = \begin{cases} \lim_{R_u\downarrow 0} \frac{0}{R_u^\beta} e^{(\beta^2-\beta)\frac{\sigma^2}{2}\Delta t} = 0 & \text{if } \beta > 1 \\ \lim_{R_u\downarrow 0} \frac{1-\beta}{R_u^\beta} e^{(\beta^2-\beta)\frac{\sigma^2}{2}\Delta t} = 0 & \text{if } \beta < 0 \\ \lim_{R_u\downarrow 0} \frac{1-\beta}{R_u^\beta} e^{(\beta^2-\beta)\frac{\sigma^2}{2}\Delta t} = \infty & \text{if } \beta \in (0,1) \\ 1 & \text{otherwise} \end{cases}$$

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